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Journal of Geometry and Physics 48 (2003) 385–425
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Jacobi groupoids and generalized Lie bialgebroids

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Received 17 December 2002

Abstract

Jacobi groupoids are introduced as a generalization of Poisson and contact groupoids and it is proved that generalized Lie bialgebroids are the infinitesimal invariants of Jacobi groupoids. Several examples are discussed.

MSC: 17B66; 22A22; 53D10; 53D17; 58H05

JGP SC: Differential geometry

Keywords: Lie groupoids; Lie algebroids; Lie bialgebroids; Poisson groupoids; Contact groupoids; Jacobi manifolds

1. Introduction

A Poisson groupoid is a Lie groupoid $G \rightrightarrows M$ with a Poisson structure Λ for which the graph of the partial multiplication is a coisotropic submanifold in the Poisson manifold $(G \times G \times G, \Lambda \oplus \Lambda \oplus -\Lambda)$ (see [40]). If $(G \rightrightarrows M, \Lambda)$ is a Poisson groupoid, then there exists a Poisson structure on M such that the source projection $\alpha : G \to M$ is a Poisson morphism. Moreover, if AG is the Lie algebroid of G, then the dual bundle A^*G to AG itself has a Lie algebroid structure. Poisson groupoids were introduced by Weinstein [40] as a generalization of both Poisson Lie groups and the symplectic groupoids which arise in the integration of arbitrary Poisson manifolds. A canonical example of symplectic groupoid is the cotangent bundle T^*G of an arbitrary Lie groupoid $G \rightrightarrows M$. In this case, the base space is A^*G and the Poisson structure on A^*G is just the linear Poisson structure induced by the Lie algebroid AG (see [2]).

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doi:10.1016/S0393-0440(03)00050-0

In [30], Mackenzie and Xu proved that a Lie groupoid $G \rightrightarrows M$ endowed with a Poisson structure Λ is a Poisson groupoid if and only if the bundle map $\#_\Lambda : T^*G \to TG$ is a morphism between the cotangent groupoid $T^*G \rightrightarrows A^*G$ and the tangent groupoid $TG \rightrightarrows TM$. This characterization was used in order to prove that Lie bialgebroids are the infinitesimal invariants of Poisson groupoids, i.e. if $(G \rightrightarrows M, \Lambda)$ is a Poisson groupoid, then (AG, A^*G) is a Lie bialgebroid and, conversely, a Lie bialgebroid structure on the Lie algebroid of a (suitably simply connected) Lie groupoid can be integrated to a Poisson groupoid structure [28,30,31] (these results can be applied to obtain a new proof of a theorem of Karasaev [19] and Weinstein [39] about the relation between symplectic groupoids and their base Poisson manifolds). We remark that in [5], Crainic and Loja Fernandes have given the precise obstructions to integrate an arbitrary Lie algebroid to a Lie groupoid.

On the other hand, a contact groupoid $(G \rightrightarrows M, \eta, \sigma)$ is a Lie groupoid $G \rightrightarrows M$ endowed with a contact 1-form $\eta \in \Omega^1(G)$ and a multiplicative function $\sigma \in C^\infty(G, \mathbb{R})$ such that

$$\eta_{(gh)}(X_g \oplus_{TG} Y_h) = \eta_g(X_g) + \mathrm{e}^{\sigma(g)}\eta_h(Y_h) \quad \text{for } (X_g, Y_h) \in TG^{(2)},$$

where $\oplus_{TG}$ is the partial multiplication in the tangent Lie groupoid $TG \rightrightarrows TM$ (see [6,7,20,23]). Contact groupoids can be considered as the odd-dimensional counterpart of symplectic groupoids and they have applications in the prequantization of Poisson manifolds and in the integration of local Lie algebras associated to rank one vector bundles (see [6,7]). In this case, the base space M carries an induced Jacobi structure such that the pair $(\alpha, \mathrm{e}^\sigma)$ is a conformal Jacobi morphism. Moreover, the presence of the multiplicative function σ induces a 1-cocycle $\phi_0 \in \Gamma(A^*G)$ in the Lie algebroid cohomology of AG. We note that the relation between Jacobi structures and Lie algebroids with 1-cocycles has been recently explored in [16] by the authors (see also [12]). More precisely, we have obtained that a Lie algebroid structure on a vector bundle $A \to M$ and a 1-cocycle $\phi_0 \in \Gamma(A^*)$, a generalized Lie algebroid in our terminology, induce a Jacobi structure $(\Lambda_{(A^*,\phi_0)} E_{(A^*,\phi_0)})$ on A^* satisfying some linearity conditions. In addition, using the differential calculus on Lie algebroids in the presence of a 1-cocycle, it has been introduced in [17] (see also [11,12]) the notion of a generalized Lie bialgebroid in such a way that a Jacobi manifold has associated a canonical generalized Lie bialgebroid. A generalized Lie bialgebroid is a pair $((A, \phi_0), (A^*, X_0))$, where (A, ϕ_0) and (A^*, X_0) are generalized Lie algebroids, such that the Lie algebroid structures on A and A^* and the 1-cocycles ϕ_0 and X_0 satisfy some compatibility conditions. When ϕ_0 and X_0 are zero, the definition reduces to that of a Lie bialgebroid. We also remark that the theory of generalized Lie algebroids plays an important role in the study of Lie brackets on affine bundles and its application in the geometrical construction of Lagrangian-type dynamics on affine bundles (see [10,32,36]).

The aim of this paper is to integrate generalized Lie bialgebroids, i.e. to introduce the notion of a Jacobi groupoid (a generalization of Poisson and contact groupoids), in terms of groupoid morphisms, such that generalized Lie bialgebroids to be considered as the infinitesimal invariants of Jacobi groupoids.

As in the case of contact groupoids, we start with a Lie groupoid $G \rightrightarrows M$, a Jacobi structure (Λ, E) on G and a multiplicative function $\sigma : G \to \mathbb{R}$. Then, as in the case of Poisson groupoids, we consider the vector bundle morphism $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ induced by the Jacobi structure (Λ, E). The multiplicative function σ induces, in a natural way, an action of the tangent groupoid $TG \rightrightarrows TM$ over the canonical projection

$\pi_1 : TM \times \mathbb{R} \to TM$ obtaining an action groupoid $TG \rightrightarrows \mathbb{R}$ over $TM \times \mathbb{R}$. Thus, it is necessary to introduce a suitable Lie groupoid structure in $T^*G \times \mathbb{R}$ over A^*G and this is the first important result of the paper. In fact, we prove that:

- If AG is the Lie algebroid of an arbitrary Lie groupoid $G \rightrightarrows M, \sigma : G \to \mathbb{R}$ is a multiplicative function, $\bar{\pi}_G : T^*G \times \mathbb{R} \to G$ is the canonical projection and η_G is the canonical contact 1-form on $T^*G \times \mathbb{R}$, then $(T^*G \times \mathbb{R} \rightrightarrows A^*G, \eta_G, \sigma \circ \bar{\pi}_G)$ is a contact groupoid in such a way that the Jacobi structure on $A * G$ is just the linear Jacobi structure $(\Lambda_{(A^*G,\phi_0)}, E_{(A^*G,\phi_0)})$ induced by the Lie algebroid AG and the 1-cocycle ϕ_0 which comes from the multiplicative function σ (see Theorems 3.7 and 3.10).

Now, we will say that $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid if the map $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a Lie groupoid morphism over some map $\varphi_0 : A^*G \to TM \times \mathbb{R}$. Poisson and contact groupoids and other interesting examples are Jacobi groupoids. In particular, Jacobi groupoids $(G \rightrightarrows M, \Lambda, E, \sigma)$, where M is a single point are just the Lie groups studied in [18], whose infinitesimal invariants are generalized Lie bialgebras.

On the other hand, if $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid, then we show that the vector bundle A^*G admits a Lie algebroid structure and the multiplicative function σ (respectively, the vector field E) induces a 1-cocycle ϕ_0 (respectively, X_0) on AG (respectively, A^*G). Thus, a first relation between Jacobi groupoids and generalized Lie bialgebroids can be obtained and this is the second important result of our paper:

- If $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid, then $((AG, \phi_0), (A^*G, X_0))$ is a generalized Lie bialgebroid (see Theorem 5.4).

  Finally, a converse of the above statement is the third important result of the paper. More precisely, we prove the following theorem.
- Let $((AG, \phi_0), (A^*G, X_0))$ be a generalized Lie bialgebroid where AG is the Lie algebroid of an α-connected and α-simply connected Lie groupoid $G \rightrightarrows M$. Then, there is a unique multiplicative function $\sigma : G \to \mathbb{R}$ and a unique Jacobi structure (Λ, E) on G that makes $(G \rightrightarrows M, \Lambda, E, \sigma)$ into a Jacobi groupoid with generalized Lie bialgebroid $((AG, \phi_0), (A^*G, X_0))$ (see Theorem 5.9).

The two previous results generalize those obtained by Mackenzie and Xu [30,31] for Poisson groupoids and those obtained by the authors [18] for generalized Lie bialgebras.

The paper is organized as follows. In Section 2, we recall several definitions and results about Jacobi manifolds, Lie algebroids and Lie groupoids which will be used in the sequel. In Section 3, we prove that a Lie groupoid $G \rightrightarrows M$ (with Lie algebroid AG) and a multiplicative function $\sigma : G \to \mathbb{R}$ induce a Lie groupoid structure in $TG \times \mathbb{R}$ over $TM \times \mathbb{R}$ and a contact groupoid structure in $T^*G \times \mathbb{R}$ over A^*G. In Section 4, we introduce the definition of a Jacobi groupoid, giving some examples, and we prove some properties of these groupoids. In Section 5, we show that generalized Lie bialgebroids are, in fact, the infinitesimal invariants of Jacobi groupoids.

Notation. If M is a differentiable manifold, we will denote by $C^\infty(M, \mathbb{R})$ the algebra of C^∞ real-valued functions on M, by $\Omega^k(M)$ the space of k-forms on M, by $\mathfrak{X}(M)$ the Lie algebra of vector fields, by δ the de Rham differential on $\Omega^*(M) = \oplus_k \Omega^k(M)$, by $\mathcal{L}$ the Lie derivative operator and by $[\,,\,]$ the Schouten–Nijenhuis bracket [1,37]. Moreover, if $A \to M$ is a vector bundle over M and $P \in \Gamma(\wedge^2 A)$ is a section of $\wedge^2 A \to M$, we will denote by

$\#_P : A^* \to A$ the bundle map given by $\nu(\#_P(\omega)) = P(x)(\omega, \nu)$ for $\omega, \nu \in A^*_x$, A^*_x being the fiber of A^* over $x \in M$. We will also denote by $\#_P : \Gamma(A^*) \to \Gamma(A)$ the corresponding homomorphism of $C^\infty(M, \mathbb{R})$-modules.

2. Jacobi structures, Lie algebroids and Lie groupoids

2.1. Jacobi structures

A *Jacobi structure* on a manifold M is a pair (Λ, E), where Λ is a 2-vector and E is a vector field on M satisfying the following properties:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \qquad [E, \Lambda] = 0. \tag{2.1}$$

The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. A bracket of functions (the *Jacobi bracket*) is defined by

$$\{f, g\} = \Lambda(\delta f, \delta g) + fE(g) - gE(f)$$

for all $f, g \in C^\infty(M, \mathbb{R})$. In fact, the space $C^\infty(M, \mathbb{R})$ endowed with the Jacobi bracket is a *local Lie algebra* in the sense of Kirillov (see [21]). Conversely, a structure of local Lie algebra on $C^\infty(M, \mathbb{R})$ defines a Jacobi structure on M (see [13,21]). If the vector field E identically vanishes, then (M, Λ) is a *Poisson manifold* (see [1,26,37,38]).

Another interesting example of Jacobi manifolds comes from contact manifolds. Let M be a $2n + 1$-dimensional manifold and η a 1-form on M. We say that (M, η) is a *contact manifold* if $\eta \wedge (\delta\eta)^n \neq 0$ at every point (see e.g. [25,27]). A contact manifold (M, η) is a Jacobi manifold whose associated Jacobi structure (Λ, E) is given by

$$\Lambda(\omega, \nu) = \delta\eta(\flat_\eta^{-1}(\omega), \flat_\eta^{-1}(\nu)), \qquad E = \flat_\eta^{-1}(\eta)$$

for $\omega, \nu \in \Omega^1(M)$, $\flat_\eta : \mathfrak{X}(M) \to \Omega^1(M)$ being the isomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\flat_\eta(X) = i(X)\delta\eta + \eta(X)\eta$. Note that E is the *Reeb vector field* of M which is characterized by the conditions $i(E)\eta = 1$ and $i(E)\delta\eta = 0$. Moreover

$$\flat_\eta^{-1}(\omega) = -\#_\Lambda(\omega) + \omega(E)E \quad \text{for } \omega \in \Omega^1(M).$$

Jacobi manifolds were introduced by Lichnerowicz [27] (see also [8,13]).

Remark 2.1. Let (Λ, E) be a 2-vector and a vector field on a manifold M. Then, we can consider the 2-vector $\tilde{\Lambda}$ given by

$$\tilde{\Lambda} = \mathrm{e}^{-t}\left(\Lambda + \frac{\partial}{\partial t} \wedge E\right), \tag{2.2}$$

where t is the usual coordinate on $\mathbb{R}$. The 2-vector $\tilde{\Lambda}$ is homogeneous with respect to the vector field $\partial/\partial t$, i.e. $\mathcal{L}_{\partial/\partial t}\tilde{\Lambda} = -\tilde{\Lambda}$. In fact, if $\tilde{\Lambda}$ is a 2-vector on $M \times \mathbb{R}$ such that $\mathcal{L}_{\partial/\partial t}\tilde{\Lambda} = -\tilde{\Lambda}$, then there exists a 2-vector Λ and a vector field E on M such that $\tilde{\Lambda}$ is given by (2.2). Moreover, (Λ, E) is a Jacobi structure on M if and only if $\tilde{\Lambda}$ defines a Poisson structure on $M \times \mathbb{R}$ (see [27]). The manifold $M \times \mathbb{R}$ endowed with the structure $\tilde{\Lambda}$ is called the *Poissonization of the Jacobi manifold* (M, Λ, E). If (Λ, E) is a Jacobi structure on M induced by a contact 1-form η, then the corresponding Poisson structure $\tilde{\Lambda}$ on $M \times \mathbb{R}$ is non-degenerate and is associated with the symplectic 2-form $\tilde{\Omega} = \mathrm{e}^t(\delta\eta + \delta t \wedge \eta)$.

Before finishing this section, we will give a definition which will be useful in the following.

Definition 2.2. Let S be a submanifold of a manifold M and Λ be an arbitrary 2-vector. S is said to be coisotropic (with respect to Λ) if $\#_\Lambda((T_xS)^\circ) \subseteq T_xS$ for $x \in S$, $(T_xS)^\circ$ being the annihilator space of T_xS.

Remark 2.3. If Λ (respectively, (Λ, E)) is a Poisson structure (respectively, a Jacobi structure) on M, then we recover the notion of a coisotropic submanifold of the Poisson manifold (M, Λ) [25,40] (respectively, coisotropic submanifold of a Jacobi manifold (M, Λ, E) [15]).

2.2. Lie algebroids

A *Lie algebroid* A over a manifold M is a vector bundle A over M together with a Lie bracket $[\![\, ,]\!]$ on the space $\Gamma(A)$ of the global cross-sections of $A \to M$ and a bundle map $\rho : A \to TM$, called the *anchor map*, such that if we also denote by $\rho : \Gamma(A) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules induced by the anchor map, then:

(i) $\rho : (\Gamma(A), [\![\, ,]\!]) \to (\mathfrak{X}(M), [\, ,])$ is a Lie algebra homomorphism and
(ii) for all $f \in C^\infty(M, \mathbb{R})$ and for all $X, Y \in \Gamma(A)$, one has

$$[\![X, fY]\!] = f[\![X, Y]\!] + (\rho(X)(f))Y.$$

The triple $(A, [\![\, ,]\!], \rho)$ is called a Lie algebroid over M (see [29,34]).

A real Lie algebra of finite dimension is a Lie algebroid over a point. Another example of a Lie algebroid is the triple $(TM, [\, ,], \mathrm{Id})$, where M is a differentiable manifold and $\mathrm{Id} : TM \to TM$ is the identity map.

If A is a Lie algebroid, the Lie bracket on $\Gamma(A)$ can be extended to the so-called *Schouten bracket* $[\![\, ,]\!]$ on the space $\Gamma(\wedge^* A) = \oplus_k \Gamma(\wedge^k A)$ of multi-sections of A in such a way that $(\oplus_k \Gamma(\wedge^k A), \wedge, [\![\, ,]\!])$ is a graded Lie algebra. In fact, the Schouten bracket satisfies the following properties:

$$[\![X, f]\!] = \rho(X)(f), \qquad [\![P, Q]\!] = (-1)^{pq}[\![Q, P]\!],$$
$$[\![P, Q \wedge R]\!] = [\![P, Q]\!] \wedge R + (-1)^{q(p+1)} Q \wedge [\![P, R]\!],$$
$$(-1)^{pr}[\![[\![P, Q]\!], R]\!] + (-1)^{qr}[\![[\![R, P]\!], Q]\!] + (-1)^{pq}[\![[\![Q, R]\!], P]\!] = 0$$

for $X \in \Gamma(A)$, $f \in C^\infty(M, \mathbb{R})$, $P \in \Gamma(\wedge^p A)$, $Q \in \Gamma(\wedge^q A)$ and $R \in \Gamma(\wedge^r A)$ (see [37]).

Remark 2.4. The definition of Schouten bracket considered here is the one given in [37] (see also [1,26]). Some authors, see e.g. [22], define the Schouten bracket in another way.

In fact, the relation between the Schouten bracket $[\![,]\!]'$ in the sense of [22] and the Schouten bracket $[\![,]\!]$ in the sense of [37] is the following one. If $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^* A)$, then $[\![P, Q]\!]' = (-1)^{p+1}[\![P, Q]\!]$.

On the other hand, imitating the de Rham differential on the space $\Omega^*(M)$, we define the *differential of the Lie algebroid A*, $\mathrm{d} : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$, as follows. For $\omega \in \Gamma(\wedge^k A^*)$ and $X_0, \ldots, X_k \in \Gamma(A)$:

$$\begin{aligned} \mathrm{d}\omega(X_0, \ldots, X_k) = {} & \sum_{i=0}^{k}(-1)^i \rho(X_i)(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) \\ & + \sum_{i<j}(-1)^{i+j}\omega([\![X_i, X_j]\!], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k). \end{aligned} \tag{2.3}$$

Moreover, since $\mathrm{d}^2 = 0$, we have the corresponding cohomology spaces. This cohomology is the *Lie algebroid cohomology with trivial coefficients* (see [29]).

Using the above definitions, it follows that a 1-cochain $\phi \in \Gamma(A^*)$ is a 1-cocycle if and only if

$$\phi[\![X, Y]\!] = \rho(X)(\phi(Y)) - \rho(Y)(\phi(X))$$

for all $X, Y \in \Gamma(A)$.

Next, we will consider some examples of Lie algebroids which will be important in the following:

1. *The Lie algebroid* $(TM \times \mathbb{R}, [,], \pi)$

   If M is a differentiable manifold, then the triple $(TM \times \mathbb{R}, [,], \pi)$ is a Lie algebroid over M, where $\pi : TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[,]$ is the bracket given by (see [29,33])

   $$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f)) \tag{2.4}$$

   for $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$.
2. *The Lie algebroid* $(T^*M \times \mathbb{R}, [\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ *associated with a Jacobi manifold* (M, Λ, E)

   A Jacobi manifold (M, Λ, E) has an associated Lie algebroid $(T^*M \times \mathbb{R}, [\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$, where $[\![,]\!]_{(\Lambda,E)}$ and $\tilde{\#}_{(\Lambda,E)}$ are defined by

   $$\begin{aligned} & [\![(\omega, f), (\nu, g)]\!]_{(\Lambda,E)} \\ & = (\mathcal{L}_{\#_\Lambda(\omega)}\nu - \mathcal{L}_{\#_\Lambda(\nu)}\omega - \delta(\Lambda(\omega, \nu)) + f\mathcal{L}_E\nu - g\mathcal{L}_E\omega - i(E)(\omega \wedge \nu), \Lambda(\nu, \omega) \\ & \quad + \#_\Lambda(\omega)(g) - \#_\Lambda(\nu)(f) + fE(g) - gE(f)), \qquad \tilde{\#}_{(\Lambda,E)}(\omega, f) = \#_\Lambda(\omega) + fE \end{aligned} \tag{2.5}$$

   for $(\omega, f), (\nu, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$, $\mathcal{L}$ being the Lie derivative operator (see [20]). In the particular case when (M, Λ) is a Poisson manifold we recover, by projection, the Lie algebroid $(T^*M, [\![,]\!]_\Lambda, \#_\Lambda)$, where $[\![,]\!]_\Lambda$ is the bracket of

1-forms defined by $[\![\omega,\nu]\!]_\Lambda = \mathcal{L}_{\#_\Lambda(\omega)}\nu - \mathcal{L}_{\#_\Lambda(\nu)}\omega - \delta(\Lambda(\omega,\nu))$ for $\omega,\nu\in\Omega^1(M)$ (see [1,2,9,37]).

3. *Action of a Lie algebroid on a smooth map*

   Let $(A,[\![,]\!],\rho)$ be a Lie algebroid over a manifold M and $\pi : P\to M$ be a smooth map. An action of A on $\pi : P\to M$ is a $\mathbb{R}$-linear map

   $$*:\Gamma(A)\to\mathfrak{X}(P),\qquad X\in\Gamma(A)\mapsto X^*\in\mathfrak{X}(P)$$

   such that

   $$(fX)^* = (f\circ\pi)X^*,\qquad [\![X,Y]\!]^* = [X^*,Y^*],\qquad \pi_*^p(X^*(p)) = \rho(X(\pi(p)))$$

   for $f\in C^\infty(M,\mathbb{R})$, $X,Y\in\Gamma(A)$ and $p\in M$. If $*:\Gamma(A)\to\mathfrak{X}(P)$ is an action of A on $\pi : P\to M$ and $\tau : A\to M$ is the bundle projection, then the pullback vector bundle of A over π:

   $$\pi^*A = \{(a,p)\in A\times P/\tau(a)=\pi(p)\}$$

   is a Lie algebroid over P with the Lie algebroid structure $([\![,]\!]_\pi,\rho_\pi)$ which is characterized by

   $$\rho_\pi(X)(p) = X^*(p),\qquad [\![X,Y]\!]_\pi = [\![X,Y]\!]\circ\pi$$

   for $X,Y\in\Gamma(A)$ and $p\in P$. The triple $(\pi^*A,[\![,]\!]_\pi,\rho_\pi)$ is called the *action Lie algebroid* of A on π and it is denoted by $A\ltimes\pi$ or $A\ltimes P$ (see [14]).

4. *The Lie algebroid associated with a linear Poisson structure*

   Let $\tau : A\to M$ be a vector bundle on a manifold M. Then, it is clear that there exists a bijection between the space $\Gamma(A^*)$ of the sections of the dual bundle $\tau^* : A^*\to M$ and the set $\mathcal{L}(A)$ of real functions on A which are linear on each fiber:

   $$\Gamma(A^*)\to\mathcal{L}(A),\qquad \nu\to\tilde{\nu}.$$

   Now, suppose that Λ is a linear Poisson structure on A with Poisson bracket $\{,\}$. This means that the Poisson bracket of two linear functions on A is again a linear function. This fact implies that the Poisson bracket of a linear function on A and a basic function is a basic function. Moreover, one may define a Lie algebroid structure $([\![,]\!],\rho)$ on $\tau^* : A^*\to M$ which is characterized by

   $$\widetilde{[\![\nu,\mu]\!]} = \{\tilde{\nu},\tilde{\mu}\},\qquad \rho(\nu)(f)\circ\tau = \{\tilde{\nu}, f\circ\tau\} \tag{2.6}$$

   for $\nu,\mu\in\Gamma(A^*)$ and $f\in C^\infty(M,\mathbb{R})$ (see [2,3]). Conversely, if A is a vector bundle over M and the dual bundle A^* admits a Lie algebroid structure $([\![,]\!],\rho)$, then one may define a linear Poisson bracket $\{,\}$ on A in such a way that (2.6) holds.

5. *The tangent Lie algebroid*

   Let (M,Λ) be a Poisson manifold. Then, the complete lift Λ^{c} of Λ to the tangent bundle *TM* defines a linear Poisson structure on *TM* (see [3,35]). Λ^{c} is called the *tangent Poisson structure*.

   Now, suppose that $\tau : A\to M$ is a Lie algebroid over a manifold M and that $p : A^*\times_M A\to\mathbb{R}$ is the natural pairing. Then, *TA* and *TA** are vector bundles over *TM* and

p induces a non-degenerate pairing $TA^* \times_{TM} TA \to \mathbb{R}$. Thus, we get an isomorphism between the vector bundles TA and $(TA^*)^*$. Therefore, the dual bundle to $TA \to TM$ may be identified with $TA^* \to TM$. On the other hand, since A^* is a Poisson manifold, we have that TA^* admits a linear Poisson structure. Consequently, the vector bundle $TA \to TM$ is a Lie algebroid which is called the *tangent Lie algebroid* to A (for more details, see [4,30]).

2.3. Lie groupoids

A *groupoid* consists of two sets G and M, called, respectively, the *groupoid* and the *base*, together with two maps α and β from G to M, called, respectively, the *source* and *target* projections, a map $\epsilon : M \to G$, called the *inclusion*, a partial multiplication $m : G^{(2)} = \{(g,h) \in G \times G/\alpha(g) = \beta(h)\} \to G$ and a map $\iota : G \to G$, called the *inversion*, satisfying the following conditions:

(i) $\alpha(m(g,h)) = \alpha(h)$ and $\beta(m(g,h)) = \beta(g)$ for all $(g,h) \in G^{(2)}$,
(ii) $m(g,m(h,k)) = m(m(g,h),k)$ for all $g,h,k \in G$ such that $\alpha(g) = \beta(h)$ and $\alpha(h) = \beta(k)$,
(iii) $\alpha(\epsilon(x)) = x$ and $\beta(\epsilon(x)) = x$ for all $x \in M$,
(iv) $m(g,\epsilon(\alpha(g))) = g$ and $m(\epsilon(\beta(g)),g) = g$ for all $g \in G$,
(v) $m(g,\iota(g)) = \epsilon(\beta(g))$ and $m(\iota(g),g) = \epsilon(\alpha(g))$ for all $g \in G$.

A groupoid G over a base M will be denoted by $G \rightrightarrows M$. Given two groupoids $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$, a *morphism of groupoids* is a pair of maps $\Phi : G_1 \to G_2$ and $\Phi_0 : M_1 \to M_2$ which commute with all the structural functions of G_1 and G_2, i.e. $\alpha_2 \circ \Phi = \Phi_0 \circ \alpha_1$, $\beta_2 \circ \Phi = \Phi_0 \circ \beta_1$ and $\Phi(g_1h_1) = \Phi(g_1)\Phi(h_1)$ for $(g_1,h_1) \in G_1^{(2)}$ (for more details, see [29]). If G and M are manifolds, $G \rightrightarrows M$ is a *Lie groupoid* if:

(i) α and β are differentiable submersions,
(ii) m, ϵ and ι are differentiable maps.

From now on, we will usually write gh for $m(g,h)$, g^{-1} for $\iota(g)$ and $\tilde{x}$ for $\epsilon(x)$. Moreover, if $x \in M$, then $G_x = \alpha^{-1}(x)$ (respectively, $G^x = \beta^{-1}(x)$) will be said the α-fiber (respectively, the β-fiber) of x. Furthermore, since ϵ is an inmersion, we will identify M with $\epsilon(M)$.

Next, we will recall some notions related with Lie groupoids which will be useful in the following (for more details, see [29]).

Definition 2.5. Let $G \rightrightarrows M$ be a Lie groupoid over a manifold M. For $U \subseteq M$ open, a local bisection (or local admissible section) of G on U is a smooth map $\mathcal{K} : U \to G$ which is right-inverse to β and for which $\alpha \circ \mathcal{K} : U \to \alpha(\mathcal{K}(U))$ is a diffeomorphism from U to the open set $\alpha(\mathcal{K}(U))$ in M. If $U = M$, $\mathcal{K}$ is a global bisection or simply a bisection.

The existence of local bisections through any point $g \in G$ is always guaranteed.

If $\mathcal{K} : U \to G$ is a local bisection with $V = (\alpha \circ \mathcal{K})(U)$, the local *left-translation* and *right-translation* induced by $\mathcal{K}$ are the maps $L_{\mathcal{K}} : \beta^{-1}(V) \to \beta^{-1}(U)$ and $R_{\mathcal{K}} : \alpha^{-1}(U) \to$

$\alpha^{-1}(V)$, defined by

$$L_{\mathcal{K}}(g) = \mathcal{K}((\alpha \circ \mathcal{K})^{-1}(\beta(g)))g, \qquad R_{\mathcal{K}}(h) = h\mathcal{K}(\alpha(h))$$

for $g \in \beta^{-1}(V)$ and $h \in \alpha^{-1}(U)$.

Remark 2.6. If $y_0 \in U$ and $\mathcal{K}(y_0) = g_0, \alpha(g_0) = x_0$, then the restriction of $L_{\mathcal{K}}$ to G^{x_0} is the left-translation by g_0:

$$L_{g_0} : G^{x_0} \to G^{y0}, \qquad h \mapsto L_{g_0}(h) = g_0 h.$$

In a similar way, the restriction of $R_{\mathcal{K}}$ to G_{y_0} is the right-translation by g_0:

$$R_{g_0} : G_{y_0} \to G_{x_0}, \qquad g \mapsto R_{g_0}(g) = gg_0.$$

A multivector field P on G is said to be *left-invariant* (respectively, *right-invariant*) if it is tangent to the fibers of β (respectively, α) and $P(gh) = (L_{\mathcal{K}})^h_*(P(h))$ (respectively, $P(gh) = (R_{\mathcal{K}})^g_*(P(g))$) for $g, h \in G$ and $\mathcal{K} : U \to G$ any local bisection through h (respectively, g). If P and Q are two left-invariant (respectively, right-invariant) multivector fields on G, then $[P, Q]$ is again left-invariant (respectively, right-invariant).

Now, we will recall the definition of the Lie algebroid associated with a Lie groupoid.

Suppose that $G \rightrightarrows M$ is a Lie groupoid. Then, we may consider the vector bundle $AG \to M$, whose fiber at a point $x \in M$ is $A_x G = T_{\tilde{x}} G^x$. It is easy to prove that there exists a bijection between the space $\Gamma(AG)$ and the set of left-invariant (respectively, right-invariant) vector fields on G. If X is a section of AG, the corresponding left-invariant (respectively, right-invariant) vector field on G will be denoted by $\overleftarrow{X}$ (respectively, $\overrightarrow{X}$). Using the above facts, we may introduce a Lie algebroid structure $([\![\,,\,]\!], \rho)$ on AG, which is defined by for $X, Y \in \Gamma(AG)$ and $x \in M$:

$$\overleftarrow{[\![X, Y]\!]} = [\overleftarrow{X}, \overleftarrow{Y}], \qquad \rho(X)(x) = \alpha^{\tilde{x}}_*(X(x)). \tag{2.7}$$

Remark 2.7. There exists a bijection between the space $\Gamma(\wedge^k(AG))$ and the set of left-invariant (respectively, right-invariant) k-vector fields. If P is a section of $\wedge^k(AG)$, we will denote by $\overleftarrow{P}$ (respectively, $\overrightarrow{P}$) the corresponding left-invariant (respectively, right-invariant) k-vector field on G. Moreover, if $P, Q \in \Gamma(\wedge^*(AG))$, we have that

$$\overleftarrow{[\![P, Q]\!]} = [\overleftarrow{P}, \overleftarrow{Q}]. \tag{2.8}$$

Example 2.8.

(1) *Lie groups*

Any Lie group G is a Lie groupoid over $\{\mathfrak{e}\}$, the identity element of G. The Lie algebroid associated with G is just the Lie algebra $\mathfrak{g}$ of G.

(2) *The banal groupoid*

Let M be a differentiable manifold. The product manifold $M \times M$ is a Lie groupoid over M in the following way: α is the projection onto the second factor and β is the projection onto the first factor; $\epsilon(x) = (x, x)$ for all $x \in M$ and $m((x, y), (y, z)) = (x, z)$. $M \times M \rightrightarrows M$ is called the *banal groupoid*. The Lie algebroid associated with the banal groupoid is the tangent bundle TM of M.

(3) *The direct product of Lie groupoids*

If $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$ are Lie groupoids, then $G_1 \times G_2 \rightrightarrows M_1 \times M_2$ is a Lie groupoid in a natural way.

(4) *Action groupoids*

Let $G \rightrightarrows M$ be a Lie groupoid and $\pi : P \to M$ be a smooth map. If $P * G = \{(p, g) \in P \times G/\pi(p) = \beta(g)\}$, then a right action of G on π is a smooth map:

$$P * G \to P, \qquad (p, g) \mapsto p \cdot g,$$

which satisfies the following relations:

$$\pi(p \cdot g) = \alpha(g) \quad \text{for all } (p, g) \in P * G,$$
$$(p \cdot g) \cdot h = p \cdot (gh) \quad \text{for all } (g, h) \in G^{(2)} \text{ and } (p, g) \in P * G,$$
$$p \cdot \widetilde{\pi(p)} = p \quad \text{for all } p \in P.$$

Given such an action one constructs the *action groupoid* $P * G \rightrightarrows P$ by defining

$$\alpha'(p, g) = p \cdot g, \quad \beta'(p, g) = p, \quad m'((p, g), (q, h)) = (p, gh) \quad \text{if } q = p \cdot g,$$
$$\epsilon'(p) = (p, \epsilon(\pi(p))), \qquad \iota'(p, g) = (p \cdot g, g^{-1}).$$

Now, if $p \in P$, we consider the map $\pi_p : G^{\pi(p)} \to P$ given by

$$\pi_p(g) = p \cdot g.$$

Then, if AG is the Lie algebroid of G, the $\mathbb{R}$-linear map:

$$* : \Gamma(AG) \to \mathfrak{X}(P), \qquad X \in \Gamma(AG) \mapsto X^* \in \mathfrak{X}(P)$$

defined by

$$X^*(p) = (\pi_p)_*^{\widetilde{\pi(p)}}(X(\pi(p))) \quad \text{for all } p \in P \tag{2.9}$$

induces an action of AG on $\pi : P \to M$. In addition, the Lie algebroid associated with the Lie groupoid $P * G \rightrightarrows P$ is the action Lie algebroid $AG \ltimes \pi$ (for more details, see [14]).

(5) *The tangent groupoid*

Let $G \rightrightarrows M$ be a Lie groupoid. Then, the tangent bundle TG is a Lie groupoid over TM. The projections α^{T}, β^{T}, the partial multiplication $\oplus_{TG}$, the inclusion ϵ^{T} and the inversion ι^{T} are defined by

$$\begin{aligned}
&\alpha^{\mathrm{T}}(X_g) = \alpha_*^g(X_g), \qquad \beta^{\mathrm{T}}(X_g) = \beta_*^g(X_g) \quad \text{for } X_g \in T_gG,\\
&X_g \oplus_{TG} Y_h = m_*^{(g,h)}(X_g, Y_h) \quad \text{for } (X_g, Y_h) \in (TG)^{(2)}_{(g,h)} = T_{(g,h)}G^{(2)},\\
&\epsilon^{\mathrm{T}}(X_x) = \epsilon_*^x(X_x) \quad \text{for } X_x \in T_xM, \qquad \iota^{\mathrm{T}}(X_g) = \iota_*^g(X_g) \quad \text{for } X_g \in T_gG.
\end{aligned} \tag{2.10}$$

In [42], it has been given an explicit expression for the multiplication $\oplus_{TG}$. If $\alpha_*^g(X_g) = \beta_*^h(X_h) = W_x$, $x = \alpha(g) = \beta(h)$, then

$$X_g \oplus_{TG} Y_h = (L_{\mathcal{X}})_*^h(Y_h) + (R_{\mathcal{Y}})_*^g(X_g) - (L_{\mathcal{X}})_*^h((R_{\mathcal{Y}})_*^{\tilde{x}}(\epsilon_*^x(W))), \tag{2.11}$$

where $\mathcal{X}$, $\mathcal{Y}$ are any (local) bisections of G with $\mathcal{X}(x) = g$ and $\mathcal{Y}(x) = h$. The tangent Lie algebroid $TAG \to TM$ is just the Lie algebroid associated with the tangent groupoid $TG \rightrightarrows TM$ (for more details, see [30]).

Remark 2.9. If G is a Lie group then, from (2.11), it follows that

$$X_g \oplus_{TG} Y_h = (L_g)_*^h(Y_h) + (R_h)_*^g(X_g) \quad \text{for } X_g \in T_gG \text{ and } Y_h \in T_hG. \tag{2.12}$$

(6) *The cotangent groupoid*

Let $G \rightrightarrows M$ be a Lie groupoid. If A^*G is the dual bundle to AG, then the cotangent bundle $T^*G$is a Lie groupoid over A^*G. The projections $\tilde{\alpha}$ and $\tilde{\beta}$, the partial multiplication $\oplus_{T^*G}$, the inclusion $\tilde{\epsilon}$ and the inversion $\tilde{\iota}$ are defined as follows:

$$\begin{aligned}
&\tilde{\alpha}(\omega_g)(X) = \omega_g((L_g)_*^{\widetilde{\alpha(g)}}(X)) \quad \text{for } \omega_g \in T_g^*G \text{ and } X \in A_{\alpha(g)}G,\\
&\tilde{\beta}(\nu_h)(Y) = \nu_h((R_h)_*^{\widetilde{\beta(h)}}(Y - \epsilon_*^{\beta(h)}(\alpha_*^{\widetilde{\beta(h)}}(Y)))) \text{ for } \nu \in T_h^*G \text{ and } Y \in A_{\beta(h)}G,\\
&(\omega_g \oplus_{T^*G} \nu_h)(X_g \oplus_{TG} Y_h) = \omega_g(X_g) + \nu_h(Y_h) \quad \text{for } (X_g, Y_h) \in T_{(g,h)}G^{(2)},\\
&\tilde{\epsilon}(\omega_x)(X_{\tilde{x}}) = \omega_x(X_{\tilde{x}} - \epsilon_*^x(\beta_*^{\tilde{x}}(X_{\tilde{x}}))) \text{ for } \omega_x \in A_x^*G, X_{\tilde{x}} \in T_{\tilde{x}} \in G \text{ and } x \in M,\\
&\tilde{\iota}(\omega_g)(X_{g^{-1}}) = -\omega_g(\iota_*^{g^{-1}}(X_{g^{-1}}))\\
&\qquad\qquad\qquad \text{for } \omega_g \in T_g^*G \text{ and } X_{g^{-1}}G \text{ and } X_{g^{-1}} \in T_{g^{-1}}G.
\end{aligned} \tag{2.13}$$

Note that $\tilde{\epsilon}(A^*G)$ is just the conormal bundle of $M \cong \epsilon(M)$ as a submanifold of G.

On the other hand, since A^*G is a Poisson manifold, the cotangent bundle $T^*(A^*G)$ is a Lie algebroid. In fact, the Lie algebroid of the cotangent Lie groupoid $T^*G \rightrightarrows A^*G$ may be identified with $T^*(A^*G)$ (for more details, see [2,30]).

Remark 2.10. If G is a Lie group and $\omega_g \in T_g^*G$, $\nu_h \in T_h^*G$ satisfy $\tilde{\alpha}(\omega_g) = \tilde{\beta}(\nu_h)$ then, from (2.12), it follows that

$$\omega_g \oplus_{T^*G} \nu_h = \tfrac{1}{2}\{((R_{h^{-1}})_*^{gh})^*(\omega_g) + ((L_{g^{-1}})_*^{gh})^*(\nu_h)\}. \tag{2.14}$$

2.4. *Generalized Lie bialgebroids*

In this section, we will recall the definition of a generalized Lie bialgebroid. First, we will exhibit some results about the differential calculus on Lie algebroids in the presence of a 1-cocycle (for more details, see [17]).

If $(A, [\![,]\!], \rho)$ is a Lie algebroid over M and, in addition, we have a 1-cocycle $\phi_0 \in \Gamma(A^*)$, then the usual representation of the Lie algebra $\Gamma(A)$ on the space $C^\infty(M, \mathbb{R})$ can be modified and a new representation is obtained. This representation is given by $\rho_{\phi_0}(X)(f) = \rho(X)(f) + \phi_0(X)f$ for $X \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{R})$. The resulting cohomology operator d_{ϕ_0} is called the ϕ_0-differential of A and its expression, in terms of the differential d of A, is

$$\mathrm{d}_{\phi_0}\omega = \mathrm{d}\omega + \phi_0 \wedge \omega \tag{2.15}$$

for $\omega \in \Gamma(\wedge^k A^*)$. The ϕ_0-differential of A allows us to define, in a natural way, the ϕ_0-Lie derivative by a section $X \in \Gamma(A)$, $(\mathcal{L}_{\phi_0})_X : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*)$, as the commutator of d_{ϕ_0} and the contraction by X, i.e. $(\mathcal{L}_{\phi_0})_X = d_{\phi_0} \circ i(X) + i(X) \circ d_{\phi_0}$ (for the general definition of the differential and the Lie derivative associated with a representation of a Lie algebroid on a vector bundle, see [29]).

On the other hand, imitating the definition of the Schouten bracket of two multilinear first-order differential operators on the space of C^∞ real-valued functions on a manifold N (see [1]), we introduced the ϕ_0-Schouten bracket of a p-section P and a p'-section P' as the $(p+p'-1)$-section given by

$$[\![P, P']\!]_{\phi_0} = [\![P, P']\!] + (-1)^{p+1}(p-1)P \wedge (i(\phi_0)P') - (p'-1)(i(\phi_0)P) \wedge P', \tag{2.16}$$

where $[\![\,,\,]\!]$ is the usual Schouten bracket of A (some properties of the ϕ_0-Schouten bracket were obtained in [17]). Moreover, using the ϕ_0-Schouten bracket, we can define the ϕ_0-Lie derivative of $P \in \Gamma(\wedge^k A)$ by $X \in \Gamma(A)$ as

$$(\mathcal{L}_{\phi_0})_X(P) = [\![X, P]\!]_{\phi_0}. \tag{2.17}$$

Remark 2.11. The product manifold $\bar{A} = A \times T\mathbb{R}$ is a vector bundle over $M \times \mathbb{R}$ and one may define a Lie algebroid structure $([\![\,,\,]\!]^{-}, \bar{\rho})$ on $\bar{A}$, where $[\![\,,\,]\!]^{-}$ is the obvious product Lie bracket and $\bar{\rho} = \rho \times \mathrm{id} : \bar{A} \to TM \times T\mathbb{R}$. The direct sum $\Gamma(\wedge^p A) \oplus \Gamma(\wedge^{p-1} A)$ is a subspace of $\Gamma(\wedge^p \bar{A})$ and we may consider the monomorphism of $C^\infty(M, \mathbb{R})$-modules $\bar{U}_{\phi_0} : \Gamma(\wedge^p A) \to \Gamma(\wedge^p \bar{A})$ given by $\bar{U}_{\phi_0}(P) = (\mathrm{e}^{-(p-1)t}P, \mathrm{e}^{-(p-1)t}i(\phi_0)(P))$. Then, it is easy to prove that $\bar{U}_{\phi_0}([\![P, P']\!]_{\phi_0}) = [\![\bar{U}_{\phi_0}(P), \bar{U}_{\phi_0}(P')]\!]^{-}$ for $P \in \Gamma(\wedge^p A)$ and $P' \in \Gamma(\wedge'^p A)$ (see [11]).

Now, suppose that $(A, [\![\,,\,]\!], \rho)$ is a Lie algebroid and that $\phi_0 \in \Gamma(A^*)$ is a 1-cocycle. Assume also that the dual bundle A^* admits a Lie algebroid structure $([\![\,,\,]\!]_*, \rho_*)$ and that $X_0 \in \Gamma(A)$ is a 1-cocycle. The pair $((A, \phi_0), (A^*, X_0))$ is a *generalized Lie bialgebroid* if

$$d_{*X_0}[\![X, Y]\!] = [\![X, d_{*X_0}Y]\!]_{\phi_0} - [\![Y, d_{*X_0}X]\!]_{\phi_0}, \qquad (\mathcal{L}_{*X_0})_{\phi_0}P + (\mathcal{L}_{\phi_0})_{X_0}P = 0 \tag{2.18}$$

for $X, Y \in \Gamma(A)$ and $P \in \Gamma(\wedge^p A)$, where d_{*X_0} (respectively, $\mathcal{L}_{*X_0}$) is the X_0-differential (respectively, the X_0-Lie derivative) of A^*. Note that the second equality in (2.18) holds if and only if

$$\phi_0(X_0) = 0, \qquad \rho(X_0) = -\rho_*(\phi_0), \qquad (\mathcal{L}_{*X_0})_{\phi_0}X + [\![X_0, X]\!] = 0 \tag{2.19}$$

for $X \in \Gamma(A)$ (see [17]). Very recently, an interesting characterization of generalized Lie bialgebroids has been obtained by Grabowski and Marmo [11] as follows. If we consider the bracket $[\![\,,\,]\!]'_{\phi_0}$ of a p-section P and a p'-section P' as the $(p+p'-1)$-section given by $[\![P, P']\!]'_{\phi_0} = (-1)^{p+1}[\![P, P']\!]_{\phi_0}$, then $((A, \phi_0), (A^*, X_0))$ is a generalized Lie bialgebroid if and only if d_{*X_0} is a derivation of $(\oplus_k \Gamma(\wedge^k A), [\![\,,\,]\!])'_{\phi_0})$, i.e.:

$$d_{*X_0}[\![P, P']\!]'_{\phi_0} = [\![d_{*X_0}P, P']\!]'_{\phi_0} + (-1)^{p+1}[\![P, d_{*X_0}P']\!]'_{\phi_0}$$

for $P \in \Gamma(\wedge^p A)$ and $P' \in \Gamma(\wedge^* A)$. In the particular case when $\phi_0 = 0$ and $X_0 = 0$, (2.18) is equivalent to the condition $\mathrm{d}_*[\![X, Y]\!] = [\![X, \mathrm{d}_* Y]\!] - [\![Y, \mathrm{d}_* X]\!]$. Thus, the pair $((A, 0), (A^*, 0))$ is a generalized Lie bialgebroid if and only if the pair (A, A^*) is a Lie bialgebroid (see [22,30]).

On the other hand, if (M, Λ, E) is a Jacobi manifold, then we proved in [17] that the pair $((TM \times \mathbb{R}, \phi_0), (T^*M \times \mathbb{R}, X_0))$ is a generalized Lie bialgebroid, where ϕ_0 and X_0 are the 1-cocycles on $TM \times \mathbb{R}$ and $T^*M \times \mathbb{R}$ given by

$$\phi_0 = (0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R}),$$
$$X_0 = (-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R}).$$

As a kind of converse, we have the following result.

Theorem 2.12 (Iglesias and Marrero [17]). *Let $((A, \phi_0), (A^*, X_0))$ be a generalized Lie bialgebroid over M. Then, the bracket of functions $\{,\}_0 : C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ given by*

$$\{f, g\}_0 := \mathrm{d}_{\phi_0} f \cdot \mathrm{d}_{*X_0} g \quad \textit{for} \quad f, g \in C^\infty(M, \mathbb{R})$$

defines a Jacobi structure on M.

If (Λ_0, E) is the Jacobi structure on M associated with the Jacobi bracket $\{,\}_0$, then

$$\#_{\Lambda_0}(\omega_0) = \rho_*(\rho^*(\omega_0)), \qquad E_0 = \rho_0(\phi_0) = -\rho(X_0) \tag{2.20}$$

for $\omega_0 \in \Omega^1(M)$, $\rho^* : \Omega^1(M) \to \Gamma(A^*)$ being the adjoint operator of the anchor map $\rho : \Gamma(A) \to \mathfrak{X}(M)$.

Next, we will recall the construction of the Lie bialgebroid associated with a generalized Lie bialgebroid (for more details, see [17]).

Let $(A, [\![,]\!], \rho)$ be a Lie algebroid over M and $\phi_0 \in \Gamma(A^*)$ be a 1-cocycle. Then, there exists two Lie algebroid structures on the vector bundle $\tilde{A} = A \times \mathbb{R} \to M \times \mathbb{R}$. First, we consider the map $* : \Gamma(A) \to \mathfrak{X}(M \times \mathbb{R})$ given by

$$X^* = \rho(X) \circ \pi_1 + (\phi_0(X) \circ \pi_1)\frac{\partial}{\partial t}, \tag{2.21}$$

where $\pi_1 : M \times \mathbb{R} \to M$ is the canonical projection onto the first factor. It is easy to prove that $*$ is an action of A on π_1 (see Section 2.2). Thus, if $\pi_1^* A$ is the pull-back of A over π_1, then the vector bundle $\pi_1^* A \to M \times \mathbb{R}$ admits a Lie algebroid structure $([\![,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$. It is clear that the vector bundles $\pi_1^* A \to M \times \mathbb{R}$ and $\tilde{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ are isomorphic and that the space of sections $\Gamma(\tilde{A})$ of $\tilde{A} \to M \times \mathbb{R}$ can be identified with the set of time-dependent sections of $A \to M$. Under this identification, the Lie algebroid structure $([\![,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$ is given by

$$[\![\tilde{X}, \tilde{Y}]\!]^{-\phi_0} = [\![\tilde{X}, \tilde{Y}]\!]^{\sim} + \phi_0(\tilde{X})\frac{\partial \tilde{Y}}{\partial t} - \phi_0(\tilde{Y})\frac{\partial \tilde{X}}{\partial t}, \qquad \bar{\rho}^{\phi_0}(\tilde{X}) = \tilde{\rho}(\tilde{X}) + \phi_0(\tilde{X})\frac{\partial}{\partial t} \tag{2.22}$$

for $\tilde{X}, \tilde{Y} \in \Gamma(\tilde{A})$, where $([\![,]\!]^{\sim}, \tilde{\rho})$ is the Lie algebroid structure on $\pi_1^* A$ defined by the zero 1-cocycle and $\partial\tilde{X}/\partial t$ (respectively, $\partial\tilde{Y}/\partial t$) denotes the derivative of $\tilde{X}$ (respectively, $\tilde{Y}$) with respect to the time.

Now, let $\Psi : \tilde{A} \to \tilde{A}$ be the isomorphism of vector bundles over the identity defined by $\Psi(v, t) = (e^t v, t)$ for $(v, t) \in A \times \mathbb{R} = \tilde{A}$. Using Ψ and the Lie algebroid structure $([\![,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$, one can introduce a new Lie algebroid structure $([\![,]\!]^{\wedge\phi_0}, \hat{\rho}^{\phi_0})$ on the vector bundle $\tilde{A} \to M \times \mathbb{R}$ in such a way that the Lie algebroids $(\tilde{A}, [\![,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$ and $(\tilde{A}, [\![,]\!]^{\wedge\phi_0}, \hat{\rho}^{\phi_0})$ are isomorphic. We have that

$$[\![\tilde{X}, \tilde{Y}]\!]^{\wedge\phi_0} = e^{-t}\left([\![\tilde{X}, \tilde{Y}]\!]^{\sim} + \phi_0(\tilde{X})\left(\frac{\partial\tilde{Y}}{\partial t} - \tilde{Y}\right) - \phi_0(\tilde{Y})\left(\frac{\partial\tilde{X}}{\partial t} - \tilde{X}\right)\right),$$
$$\hat{\rho}^{\phi_0}(\tilde{X}) = e^{-t}\left(\tilde{\rho}(\tilde{X}) + \phi_0(\tilde{X})\frac{\partial}{\partial t}\right) \tag{2.23}$$

for all $\tilde{X}, \tilde{Y} \in \Gamma(\tilde{A})$. Moreover, one may prove the following result.

Theorem 2.13 (Iglesias and Marrero [17]). *Let $((A, \phi_0), (A^*, X_0))$ be a generalized Lie bialgebroid and (Λ, E) be the induced Jacobi structure on M. Consider on $\tilde{A} = A \times \mathbb{R}$ (respectively, $\tilde{A}^* = A^* \times \mathbb{R}$) the Lie algebroid structure $([\![,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$ (respectively, $([\![,]\!]_*^{\wedge X_0}, \hat{\rho}_*^{X_0})$). Then*:

- (i) *The pair $(\tilde{A}, \tilde{A}^*)$ is a Lie bialgebroid over $M \times \mathbb{R}$.*
- (ii) *If $\tilde{\Lambda}$ is the induced Poisson structure on $M \times \mathbb{R}$, then $\tilde{\Lambda}$ is the Poissonization of the Jacobi structure (Λ, E).*

3. Contact groupoids and 1-jet Lie groupoids

First, we will recall the notion of a contact groupoid.

Definition 3.1 (Kerbrat and Souici-Benhammadi [20]). Let $G \rightrightarrows M$ be a Lie groupoid, $\eta \in \Omega^1(G)$ be a contact 1-form on G and $\sigma : G \to \mathbb{R}$ be an arbitrary function. If $\oplus_{TG}$ is the partial multiplication in the Lie groupoid $TG \rightrightarrows TM$, we will say that $(G \rightrightarrows M, \eta, \sigma)$ is a contact groupoid if and only if

$$\eta_{gh}(X_g \oplus_{TG} Y_h) = \eta_g(X_g) + e^{\sigma(g)}\eta_h(Y_h) \quad \text{for } (X_g, Y_g) \in TG^{(2)}. \tag{3.1}$$

Remark 3.2. Actually, the definition of a contact groupoid given in [20] is slightly different to the one given here. The relation between both approaches is the following one. If $(G \rightrightarrows M, \theta, \kappa)$ is a contact groupoid in the sense of Kerbrat and Souici-Benhammadi [20], then $(G \rightrightarrows M, \eta, \sigma)$ is a contact groupoid in the sense of Definition 3.1, where $\sigma(g) = \kappa(g^{-1})$ for $g \in G$, and η_g is the inverse of $\theta_{g^{-1}}$ in the Lie groupoid $T^* \rightrightarrows A^*G$.

If $(G \rightrightarrows M, \eta, \sigma)$ is a contact groupoid then, using the associativity of $\oplus_{TG}$, we deduce that $\sigma : G \to \mathbb{R}$ is a multiplicative function, i.e.:

$$\sigma(gh) = \sigma(g) + \sigma(h) \tag{3.2}$$

for $(g,h) \in G^{(2)}$. In particular, $\sigma_{|\epsilon(M)} = 0$ and therefore, using (3.1), it follows that $\eta_{\tilde{x}}(\epsilon_*^x(X_x)) = 0$ for $x \in M$ and $X_x \in T_xM$. Thus, if $\iota : G \to G$ is the inversion of G, we obtain that $\iota^*\eta = -\mathrm{e}^{-\sigma}\eta$. This implies that G is a contact groupoid in the sense of [7]. Using this fact, we deduce the following result.

Proposition 3.3. *Let* $(G \rightrightarrows M, \eta, \sigma)$ *be a contact groupoid and suppose that* $\dim G = 2n+1$. *Then*:

(i) *If g and h are composable elements of G, we have that*

$$(\delta\eta)_{gh}(X_g \oplus_{TG} Y_h, X'_g \oplus_{TG} Y'_h) = (\delta\eta)_g(X_g, X'_g) + \mathrm{e}^{\sigma(g)}(\delta\eta)_h(Y_h, Y'_h) + \mathrm{e}^{\sigma(g)}(X_g(\sigma)\eta_h(Y'_h) - X'_g(\sigma)\eta_h(Y_h)) \quad (3.3)$$

for $(X_g, Y_h), (X'_g, Y'_h) \in TG^{(2)}$.

(ii) $M \cong \epsilon(M)$ *is a Legendre submanifold of G, i.e.* $\epsilon^*\eta = 0$ *and* $\dim \epsilon(M) = \dim M = n$.

(iii) *If* (Λ, E) *is the Jacobi structure associated with the contact* 1*-form* η, *then E is a right-invariant vector field on G and* $E(\sigma) = 0$. *Moreover, if* $X_0 \in \Gamma(AG)$ *is the section of the Lie algebroid AG of G satisfying* $E = -\overrightarrow{X}_0$, *we have that*

$$\#_\Lambda(\delta\sigma) = \overrightarrow{X}_0 - \mathrm{e}^{-\sigma}\overleftarrow{X}_0. \quad (3.4)$$

(iv) *If* $\alpha^{\mathrm{T}}, \beta^{\mathrm{T}}$ *and* ϵ^{T} (*respectively,* $\tilde{\alpha}, \tilde{\beta}$ *and* $\tilde{\epsilon}$) *are the projections and the inclusion in the Lie groupoid* $TG \rightrightarrows TM$ (*respectively,* $T^*G \rightrightarrows A^*G$), *then*

$$\mathrm{e}^{-\sigma}\#_\Lambda \circ \tilde{\epsilon} \circ \tilde{\alpha} = \epsilon^{\mathrm{T}} \circ \alpha^{\mathrm{T}} \circ \#_\Lambda, \qquad \#_\Lambda \circ \tilde{\epsilon} \circ \tilde{\beta} = \epsilon^{\mathrm{T}} \circ \beta^{\mathrm{T}} \circ \#_\Lambda.$$

Proof. Using the results in [7], we directly deduce (i), (ii) and (iii).

Now, we will prove (iv). Suppose that $\omega_g \in T_g^*G$. Then, from (ii), we conclude that

$$\eta_{\widetilde{\alpha(g)}}(\mathrm{e}^{-\sigma(g)}\#_\Lambda(\tilde{\epsilon}(\tilde{\alpha}(\omega_g)))) = \eta_{\widetilde{\alpha(g)}}(\epsilon_*^{\alpha(g)}(\alpha_*^g(\#_\Lambda(\omega_g)))) = 0.$$

Furthermore, if $X_{\alpha(g)} \in A_{\alpha(g)}G$, it follows that

$$\epsilon_*^{\alpha(g)}(\alpha_*^g(\#_\Lambda(\omega_g))) = \iota_*^g(\#_\Lambda(\omega_g)) \oplus_{TG} \#_\Lambda(\omega_g),$$
$$X_{\alpha(g)} = 0_{T_{g^{-1}}G} \oplus_{TG} (L_g)_*^{\widetilde{\alpha(g)}}(X_{\alpha(g)})$$

and consequently, using (2.13), (3.1), (3.3) and (3.4) and the fact that σ is a multiplicative function, we obtain that

$$(\delta\eta)_{\widetilde{\alpha(g)}}(\epsilon_*^{\alpha(g)}(\alpha_*^g(\#_\Lambda(\omega_g))), X_{\alpha(g)}) = (\delta\eta)_{\widetilde{\alpha(g)}}(\mathrm{e}^{-\sigma(g)}\#_\Lambda(\tilde{\epsilon}(\tilde{\alpha}(\omega_g))), X_{\alpha(g)}).$$

On the other hand, from (2.13) and (ii), we deduce that

$$(\delta\eta)_{\widetilde{\alpha(g)}}(\epsilon_*^{\alpha(g)}(\alpha_*^g(\#_\Lambda(\omega_g))), \epsilon_*^{\alpha(g)}(Y_{\alpha(g)})) = (\delta\eta)_{\widetilde{\alpha(g)}}(\mathrm{e}^{-\sigma(g)}\#_\Lambda(\tilde{\epsilon}(\tilde{\alpha}(\omega_g))), \epsilon_*^{\alpha(g)}(Y_{\alpha(g)})) = 0$$

for $Y_{\alpha(g)} \in T_{\alpha(g)}M$.

The above facts imply that $\epsilon^{\mathrm{T}}(\alpha^{\mathrm{T}}(\#_{\Lambda}(\omega_g))) = \mathrm{e}^{-\sigma(g)}\#_{\Lambda}(\tilde{\epsilon}(\tilde{\alpha}(\omega_g)))$. In a similar way, one may prove that $\#_{\Lambda}(\tilde{\epsilon}(\tilde{\beta}(\omega_g))) = \epsilon^{\mathrm{T}}(\beta^{\mathrm{T}}(\#_{\Lambda}(\omega_g)))$. $\square$

Using again the results in [7], we have the following proposition.

Proposition 3.4. *Let* $(G \rightrightarrows M, \eta, \sigma)$ *be a contact groupoid and* $\mathfrak{X}_{\mathrm{L}}(G)$ *be the set of left-invariant vector fields on G. Denote by* (Λ, E) *the Jacobi structure on G associated with the contact* 1*-form* η, *by* $X_0 \in \Gamma(AG)$ *the section of the Lie algebroid AG of G satisfying* $E = -\overrightarrow{X}_0$ *and by* $\mathcal{I} : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \mathfrak{X}(G)$ *the map defined by*

$$\mathcal{I}(\omega_0, f_0) = \#_{\Lambda}(\mathrm{e}^{\sigma}\alpha^*\omega_0) - (\alpha^* f_0)\overleftarrow{X}_0. \tag{3.5}$$

Then:

(i) $\mathcal{I}$ *defines an isomorphism of* $C^\infty(M, \mathbb{R})$ *modules between the spaces* $\Omega^1(M) \times C^\infty(M, \mathbb{R})$ *and* $\mathfrak{X}_{\mathrm{L}}(G)$.

(ii) *The base manifold M admits a Jacobi structure* (Λ_0, E_0) *in such a way that the projection* β *is a Jacobi antimorphism and the pair* $(\alpha, \mathrm{e}^{\sigma})$ *is a conformal Jacobi morphism, i.e.*:

$$\Lambda_0(\alpha(g)) = \mathrm{e}^{\sigma(g)}\alpha_*^g(\Lambda(g)), \qquad E_0(\alpha(g)) = \alpha_*^g(X_{\mathrm{e}^{\sigma}}(g)),$$
$$\Lambda(\beta(g)) = -\beta_*^g(\Lambda_0(g)), \qquad E_0(\beta(g)) = -\beta_*^g(E(g)) \tag{3.6}$$

for all $g \in G$, *where* $X_{\mathrm{e}^{\sigma}} = \mathrm{e}^{\sigma}\#_{\Lambda}(\delta\sigma) + \mathrm{e}^{\sigma}E$ *is the Hamiltonian vector field of the function* e^{σ} *with respect to the Jacobi structure* (Λ, E).

(iii) *The map* $\mathcal{I}$ *induces an isomorphism between the Lie algebroids* $(T^*M \times \mathbb{R}, [\![,]\!]_{(\Lambda_0,E_0)}$ $\tilde{\#}_{(\Lambda_0,E_0)})$ *and AG.*

Remark 3.5. Denote also by $\mathcal{I} : T^*M \times \mathbb{R} \to AG$ *the Lie algebroid isomorphism induced by the isomorphism of* $C^\infty(M, \mathbb{R})$-modules $\mathcal{I} : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \mathfrak{X}_{\mathrm{L}}(G)$. Then, from (3.5) and since σ is a multiplicative function, it follows that

$$\mathcal{I}(\omega_x, \gamma) = \#_{\Lambda}((\alpha_*^{\tilde{x}})^*(\omega_x)) - \gamma X_0(x) \tag{3.7}$$

for $(\omega_x, \gamma) \in T_x^*M \times \mathbb{R}$.

Now, let $G \rightrightarrows M$ be a Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, there exists a natural right action of the tangent groupoid $TG \rightrightarrows TM$ on the projection $\pi_1 : TM \times \mathbb{R} \to TM$ given by

$$(v_x, \lambda) \cdot X_g = (v_x, X_g(\sigma) + \lambda)$$

for $(v_x, \lambda) \in TM \times \mathbb{R}$ and $X_g \in T_gG$ satisfying $\beta^{\mathrm{T}}(X_g) = \pi_1(v_x, \lambda)$ (see Example 2.8). The resulting action groupoid is isomorphic to $TG \times \mathbb{R} \rightrightarrows TM \times \mathbb{R}$ with projections $(\alpha^{\mathrm{T}})_\sigma$, $(\beta^{\mathrm{T}})_\sigma$, partial multiplication $\oplus_{TG\times\mathbb{R}}$, inclusion $(\epsilon^{\mathrm{T}})_\sigma$ and inversion $(\iota^{\mathrm{T}})_\sigma$ given by

$$(\alpha^{\mathrm{T}})_\sigma(X_g, \lambda) = (\alpha^{\mathrm{T}}(X_g), X_g(\sigma) + \lambda) \quad \text{for } (X_g, \lambda) \in T_gG \times \mathbb{R},$$
$$(\beta^{\mathrm{T}})_\sigma(T_h, \mu) = (\beta^{\mathrm{T}}(Y_h), \mu) \quad \text{for } (Y_h, \mu) \in T_hG \times \mathbb{R},$$
$$(X_g, \lambda) \oplus_{TG\times\mathbb{R}} (Y_h, \mu) = (X_g \oplus_{TG} Y_h, \lambda) \quad \text{if } (\alpha^{\mathrm{T}})_\sigma(X_g, \lambda) = (\beta^{\mathrm{T}})_\sigma(Y_h, \mu),$$

$$(\epsilon^{\mathrm{T}})_\sigma(X_x,\lambda)=(\epsilon^{\mathrm{T}}(X_x),\lambda)\quad \text{for } (X_x,\lambda)\in T_xM\times\mathbb{R},$$
$$(\iota^{\mathrm{T}})_\sigma(X_g,\lambda)=(\iota^{\mathrm{T}}(X_g),X_g(\sigma)+\lambda)\quad \text{for } (X_g,\lambda)\in T_gG\times\mathbb{R}. \tag{3.8}$$

Now, suppose that $(G\rightrightarrows M,\eta,\sigma)$ is a contact groupoid. Since η is a contact 1-form, the map $\#_{(\delta\eta,\eta)}:TG\times\mathbb{R}\to T^*G\times\mathbb{R}$ given by

$$\#_{(\delta\eta,\eta)}(X_g,\lambda)=(-i(X_g)(\delta\eta)_g-\lambda\eta_g,\eta_g(X_g)) \tag{3.9}$$

is an isomorphism of vector bundles. The inverse map of $\#_{(\delta\eta,\eta)}$ is the homomorphism $\#_{(\Lambda,E)}:T^*G\times\mathbb{R}\to TG\times\mathbb{R}$ defined by

$$\#_{(\Lambda,E)}(\omega_g,\gamma)=(\#_\Lambda(\omega_g)+\gamma E(g),-\omega_g(E(g))), \tag{3.10}$$

where (Λ,E) is the Jacobi structure associated with the contact 1-form η.

On the other hand, if A^*G is the dual bundle to the Lie algebroid AG then, since $\epsilon(M)$ is a Legendre submanifold of G, the map $\psi_0:TM\times\mathbb{R}\to A^*G$ given by

$$\psi_0(X_x,\lambda)=(-i(\epsilon_x^*(X_x))(\delta\eta)_{\tilde{x}}-\lambda\eta_{\tilde{x}})_{|A_xG}\quad \text{for } (X_x,\lambda)\in T_xM\times\mathbb{R} \tag{3.11}$$

is an isomorphism of vector bundles. Note that $\#_{(\delta\eta,\eta)}(\epsilon_*^x(X_x),\lambda)=(\tilde{\epsilon}(\psi_0(X_x,\lambda)),0)$ and thus the inverse map $\varphi_0:A^*G\to TM\times\mathbb{R}$ of ψ_0 is defined by

$$\varphi_0(\omega_x)=(\alpha_*^{\tilde{x}}(\#_\Lambda(\tilde{\epsilon}(\omega_x))),-\omega_x(E_{\tilde{x}}-\epsilon_*^x(\beta_*^{\tilde{x}}(E_{\tilde{x}})))), \tag{3.12}$$

$\tilde{\epsilon}:A^*G\to T^*G$ being the inclusion of identities in the Lie groupoid $T^*G\rightrightarrows A^*G$.

Next, we consider the maps $\tilde{\alpha}_\sigma,\tilde{\beta}_\sigma:T^*G\times\mathbb{R}\to A^*G$, $\tilde{\epsilon}_\sigma:A^*G\to T^*G\times\mathbb{R}$ and $\tilde{\iota}_\sigma:T^*G\times\mathbb{R}\to T^*G\times\mathbb{R}$ given by

$$\tilde{\alpha}_\sigma=\psi_0\circ(\alpha^{\mathrm{T}})_\sigma\circ\#_{(\Lambda,E)},\qquad \tilde{\beta}_\sigma=\psi_0\circ(\beta^{\mathrm{T}})_\sigma\circ\#_{\Lambda,E},$$
$$\tilde{\epsilon}_\sigma=\#_{(\delta\eta,\eta)}\circ(\epsilon^{\mathrm{T}})_\sigma\circ\varphi_0,\qquad \tilde{\iota}_\sigma=\#_{(\delta\eta,\eta)}\circ(\iota^{\mathrm{T}})_\sigma\circ\#_{(\Lambda,E)} \tag{3.13}$$

and the partial multiplication $\oplus_{T^*G\times\mathbb{R}}$ defined as follows. If $(\omega_g,\gamma),(\nu_h,\zeta)\in T^*G\times\mathbb{R}$ satisfy $\tilde{\alpha}_\sigma(\omega_g,\gamma)=\tilde{\beta}_\sigma(\nu_h,\zeta)$, then

$$(\omega_g,\gamma)\oplus_{T^*G\times\mathbb{R}}(\nu_h,\zeta)=\#_{(\delta\eta,\eta)}(\#_{(\Lambda,E)}(\omega_g,\gamma)\oplus_{TG\times\mathbb{R}}\#_{(\Lambda,E)}(\nu_h,\zeta)). \tag{3.14}$$

It is clear $\tilde{\alpha}_\sigma,\tilde{\beta}_\sigma,\tilde{\epsilon}_\sigma,\tilde{\iota}_\sigma$ and the partial multiplication $\oplus_{T^*G\times\mathbb{R}}$ are the structural functions of a Lie groupoid structure in $T^*G\times\mathbb{R}$ over A^*G. In addition, the map $\#_{(\Lambda,E)}:T^*G\times\mathbb{R}\to TG\times\mathbb{R}$ is a Lie groupoid isomorphism over $\varphi_0:A^*G\to TM\times\mathbb{R}$.

Lemma 3.6. *If $\tilde{\alpha},\tilde{\beta},\oplus_{T^*G},\tilde{\epsilon}$ and $\tilde{\iota}$ are the structural functions of the Lie groupoid $T^*G\rightrightarrows A^*G$, we have that*

$$\begin{aligned}
&\tilde{\alpha}_\sigma(\omega_g,\gamma)=\mathrm{e}^{-\sigma(g)}\tilde{\alpha}(\omega_g)\quad \textit{for } (\omega_g,\gamma)\in T_g^*G\times\mathbb{R},\\
&\tilde{\beta}_\sigma(\nu_h,\zeta)=\tilde{\beta}(\nu_h)-\zeta(\delta\sigma)_{\widetilde{\beta(h)}|A_{\beta(h)}G}\quad \textit{for } (\nu_h,\zeta)\in T_h^*G\times\mathbb{R},\\
&((\omega_g,\gamma)\oplus_{T^*G\times\mathbb{R}}(\nu_h,\zeta))=((\omega_g+\mathrm{e}^{\sigma(g)}\zeta(\delta\sigma)_g)\oplus_{T^*G}(\mathrm{e}^{\sigma(g)}\nu_h),\gamma+\mathrm{e}^{\sigma(g)}\zeta),\\
&\tilde{\epsilon}_\sigma(\omega_x)=(\tilde{\epsilon}(\omega_x),0)\quad \textit{for } \omega_x\in A_x^*G,\\
&\tilde{\iota}_\sigma(\omega_g,\gamma)=(\mathrm{e}^{-\sigma(g)}(\tilde{\iota}(\omega_g)-\gamma(\delta\sigma)_{g^{-1}}),-\mathrm{e}^{-\sigma(g)}\gamma)\quad \textit{for } (\omega_g,\gamma)\in T_g^*G\times\mathbb{R}.
\end{aligned} \tag{3.15}$$

Proof. A long computation, using (2.13), (3.1), (3.2), (3.8)–(3.14) and Proposition 3.3, proves the result. □

Note that the maps $\tilde{\alpha}_\sigma, \tilde{\beta}_\sigma, \tilde{\epsilon}_\sigma, \tilde{\iota}_\sigma$, and the partial multiplication $\oplus_{T^*G\times\mathbb{R}}$ do not depend on the contact 1-form η. In fact, one may prove the following result.

Theorem 3.7. *Let $G \rightrightarrows M$ be an arbitrary Lie groupoid with Lie algebroid AG and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then*:

(i) *The product manifold $T^*G \times \mathbb{R}$ admits a Lie groupoid structure over A^*G with structural functions given by (3.15).*
(ii) *If η_G is the canonical contact 1-form on $T^*G \times \mathbb{R}$ and $\bar{\pi}_G : T^*G \times \mathbb{R} \to G$ is the canonical projection, then $(T^*G \times \mathbb{R} \rightrightarrows A^*G, \eta_G, \sigma \circ \bar{\pi}_G)$ is a contact groupoid.*

Proof. Since σ is a multiplicative function, we obtain that

$$\epsilon^*\sigma = 0. \tag{3.16}$$

Moreover, if $(g,h) \in G^{(2)}$ and $\alpha(g) = \beta(h) = x \in M$ then, from (2.13), it follows that

$$\tilde{\alpha}((\delta\sigma)_g) = \tilde{\beta}((\delta\sigma)_h) = (\delta\sigma)_{\tilde{x}|A_xG}, \qquad (\delta\sigma)_{gh} = (\delta\sigma)_g \oplus_{T^*G} (\delta\sigma)_h. \tag{3.17}$$

In addition, using again (2.13) and the fact that σ is a multiplicative function, we have that

$$\tilde{\epsilon}((\delta\sigma)_{\tilde{x}|A_xG}) = (\delta\sigma)_{\tilde{x}}, \qquad \tilde{\iota}((\delta\sigma)_g) = (\delta\sigma)_{g^{-1}} \tag{3.18}$$

for $x \in M$ and $g \in G$.

Thus, from (3.15)–(3.18), we deduce (i).

Now, let $G \times \mathbb{R} \rightrightarrows M$ be the semi-direct Lie groupoid with projections α', β' partial multiplication m', inclusion ϵ' and inversion ι' defined by

$$\begin{aligned} &\alpha'(g,\gamma) = \alpha(g), \qquad \beta'(g,\gamma) = \beta(g) \quad \text{for } (g,\gamma) \in G \times \mathbb{R},\\ &m'((g,\gamma),(h,\zeta)) = (gh, \gamma + \mathrm{e}^{\sigma(g)}\zeta) \quad \text{for } ((g,\gamma),(h,\zeta)) \in (G\times\mathbb{R})^{(2)},\\ &\epsilon'(x) = (\epsilon(x), 0) \quad \text{for } x \in M,\\ &\iota'(g,\gamma) = (\iota(g), -\mathrm{e}^{-\sigma(g)}\gamma) \quad \text{for } (g,\gamma) \in G \times \mathbb{R}. \end{aligned} \tag{3.19}$$

Using (3.19), one may prove that the partial multiplication $\oplus_{T(G\times\mathbb{R})}$ in the tangent Lie groupoid $T(G \times \mathbb{R}) \rightrightarrows TM$ is given by

$$\begin{aligned} &\left(X_g + \psi \frac{\partial}{\partial t|_\gamma}\right) \oplus_{T(G\times\mathbb{R})} \left(Y_h + \varphi \frac{\partial}{\partial t|_\zeta}\right)\\ &\quad = (X_g \oplus_{TG} Y_h) + (\psi + \mathrm{e}^{\sigma(g)}(\zeta X_g(\sigma) + \varphi)) \frac{\partial}{\partial t|_{\gamma + \mathrm{e}^{\sigma(g)}\zeta}}. \end{aligned} \tag{3.20}$$

Next, we consider the map $\tilde{\pi}_G : T^*G \times \mathbb{R} \to G \times \mathbb{R}$ given by

$$\tilde{\pi}_G(\omega_g, \gamma) = (\pi_G(\omega_g), \gamma) \quad \text{for } (\omega_g, \gamma) \in T^*_gG \times \mathbb{R},$$

where $\pi_G : T^*G \to G$ is the canonical projection. From (3.15) and (3.19), we deduce that $\tilde{\pi}_G$ is a Lie groupoid morphism over the map $\tilde{\pi}_0 : A^*G \to M$ defined by

$$\tilde{\pi}_0(\omega_x) = x \quad \text{for} \ \ \omega_x \in A^*_x G.$$

Therefore, the tangent map to $\tilde{\pi}_G$, $T\tilde{\pi}_G : T(T^*G \times \mathbb{R}) \to T(G \times \mathbb{R})$, given by

$$T\tilde{\pi}_G \left(X_{\omega_g} + \psi \frac{\partial}{\partial t|_\gamma} \right) = (\pi_G)_*^{\omega_g}(X_{\omega_g}) + \psi \frac{\partial}{\partial t|_\gamma} \tag{3.21}$$

for $X_{\omega_g} + \psi(\partial/\partial t)|_\gamma \in T_{(\omega_g,\gamma)}(G \times \mathbb{R})$ is also a Lie groupoid morphism (over the map $T\tilde{\pi}_0 : T(A^*G) \to TM$) between the tangent Lie groupoids $T(T^*G \times \mathbb{R}) \rightrightarrows T(A^*G)$ and $T(G \times \mathbb{R}) \rightrightarrows TM$.

On the other hand, if η_G is the canonical contact 1-form on $T^*G \times \mathbb{R}$, then $\eta_G = \lambda_G - \delta t$, λ_G being the Liouville 1-form on T^*G and (see (3.21))

$$\begin{aligned}\eta_{G(\omega_g,\lambda)} \left(X_{\omega_g} + \psi \frac{\partial}{\partial t|_\gamma} \right) &= \lambda_{G(\omega_g)}(X_{\omega_g}) - \delta t|_\gamma \left(\psi \frac{\partial}{\partial t|_\gamma} \right) = \omega_g((\pi_G)_*^{\omega_g}(X_{\omega_g})) - \psi \\ &= (\omega_g - \delta t|_\gamma) \left(T\tilde{\pi}_G \left(X_{\omega_g} + \psi \frac{\partial}{\partial t|_\gamma} \right) \right). \end{aligned} \tag{3.22}$$

Thus, using (3.15), (3.20)–(3.22) and the fact that $T\tilde{\pi}_G$ is a Lie groupoid morphism, we conclude that

$$\eta_{G((\omega_g,\gamma) \oplus_{T^*G\times\mathbb{R}} (\nu_h,\zeta))} = \eta_{G(\omega_g,\gamma)} \oplus_{T^*(T^*G\times\mathbb{R})} (\mathrm{e}^{\sigma(g)} \eta_{G(\nu_h,\zeta)}),$$

i.e. $(T^*G \times \mathbb{R} \rightrightarrows A^*G, \eta_G, \bar{\sigma})$ is a contact groupoid, where $\bar{\sigma} \in C^\infty(T^*G \times \mathbb{R})$ is the function given by $\bar{\sigma} = \sigma \circ \bar{\pi}_G$. □

Remark 3.8. Let $G \rightrightarrows M$ be a Lie groupoid, $\sigma : G \to \mathbb{R}$ be a multiplicative function and $TG \times \mathbb{R} \rightrightarrows TM \times \mathbb{R}$, $T^*G \times \mathbb{R} \rightrightarrows A^*G$ be the corresponding Lie groupoids with structural functions given by (3.8) and (3.15). If σ identically vanishes then we recover, by projection, the Lie groupoids $TG \rightrightarrows TM$ and $T^*G \rightrightarrows A^*G$ (see Example 2.8).

Remark 3.9.

(i) A Lie groupoid $G \rightrightarrows M$ is said to be symplectic if G admits a symplectic 2-form Ω in such a way that the graph of the partial multiplication in G is a Lagrangian submanifold of the symplectic manifold $(G \times G \times G, \Omega \oplus \Omega \oplus (-\Omega))$ (see [2]). If $G \rightrightarrows M$ is an arbitrary Lie groupoid with Lie algebroid AG and on the cotangent Lie groupoid T^*G we consider the canonical symplectic 2-form $\Omega_G = -\delta\lambda_G$, then T^*G is a symplectic groupoid over A^*G (see [2]).

(ii) Let $G \rightrightarrows M$ be a symplectic groupoid with exact symplectic 2-form $\Omega = -\delta\theta$. Then, since $\mathbb{R}$ is a Lie group, the product manifold $G \times \mathbb{R}$ is a Lie groupoid over M (see Examples 2.8, 3). In addition, $(G \times \mathbb{R} \rightrightarrows M, \eta, 0)$ is a contact groupoid, where η is the 1-form on $G \times \mathbb{R}$ given by $\eta = \pi_1^*(\theta) - \pi_2^*(\delta t)$ and $\pi_1 : G \times \mathbb{R} \to G$, $\pi_2 : G \times \mathbb{R} \to \mathbb{R}$ are the canonical projections (see [24]). In particular, if $G \rightrightarrows M$ is an arbitrary Lie groupoid with Lie algebroid AG, then we have that $(T^*G \times \mathbb{R} \rightrightarrows A^*G, \eta_G, 0)$ is a contact groupoid, η_G being the canonical contact 1-form on $T^*G \times \mathbb{R}$. Note that, using Theorem 3.7, we directly deduce this result.

Let $G \rightrightarrows M$ be an arbitrary Lie groupoid with Lie algebroid AG and $\sigma : G \to \mathbb{R}$ be a multiplicative function. From Proposition 3.4, it follows that the contact groupoid structure on $T^*G \times \mathbb{R}$ induces a Jacobi structure on the vector bundle A^*G. Next, we will describe such a Jacobi structure. For this purpose, we will recall the definition of the linear Jacobi structure associated with a Lie algebroid and a 1-cocycle on it (for more details, see [16]).

Suppose that $(L, [\![,]\!], \rho)$ is a Lie algebroid over M and denote by Λ_{L^*} the corresponding linear Poisson structure on L^* (see Section 2.2). If $\omega_0 \in \Gamma(L^*)$ is a 1-cocycle of L, Δ is the Liouville vector field of L^* and $\omega_0^v \in \mathfrak{X}(L^*)$ is the vertical lift of ω_0, we have that the pair $(\Lambda_{(L^*,\omega_0)}, E_{(L^*,\omega_0)})$ is a Jacobi structure on L^*, where

$$\Lambda_{(L^*,\omega_0)} = \Lambda_{L^*} + \Delta \wedge \omega_0^v, \qquad E_{(L^*,\omega_0)} = -\omega_0^v. \tag{3.23}$$

The Jacobi bracket $\{,\}_{(L^*,\omega_0)}$ associated with the Jacobi structure $(\Lambda_{(L^*,\omega_0)}, E_{(L^*,\omega_0)})$ is characterized by the following conditions:

$$\{\tilde{X}, \tilde{Y}\}_{(L^*,\omega_0)} = \widetilde{[\![X, Y]\!]}, \qquad \{\tilde{X}, 1\}_{(L^*,\omega_0)} = \omega_0(X) \circ \tau^* \tag{3.24}$$

for $X, Y \in \Gamma(L)$, $\tau^* : L^* \to M$ being the bundle projection. Here, if $Z \in \Gamma(L)$, we denote by $\tilde{Z}$ the corresponding linear function on L^* (see [16]).

Theorem 3.10. *Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid AG and $\sigma : G \to \mathbb{R}$ be a multiplicative function. If $\bar{\pi}_G : T^*G \times \mathbb{R} \to G$ is the canonical projection, η_G is the canonical contact* 1*-form on $T^*G \times \mathbb{R}$ and (Λ_0, E_0) is the Jacobi structure on A^*G induced by the contact groupoid $(T^*G \times \mathbb{R} \rightrightarrows A^*G, \eta_G, \bar{\sigma} = \sigma \circ \bar{\pi}_G)$, then*

$$\Lambda_0 = \Lambda_{(A^*G,\phi_0)}, \qquad E_0 = E_{(A^*G,\phi_0)}, \tag{3.25}$$

where $\phi_0 \in \Gamma(A^*G)$ is the 1*-cocycle of the Lie algebroid AG defined by*

$$\phi_0(x)(X_x) = X_x(\sigma) \quad \text{for } x \in M \text{ and } X_x \in A_xG. \tag{3.26}$$

Proof. Denote by $\pi_1 : T^*G \times \mathbb{R} \to T^*G$ the canonical projection onto the first factor. It is easy to prove that π_1 is a Jacobi morphism between the contact manifold $(T^*G \times \mathbb{R}, \eta_G)$ and the symplectic manifold (T^*G, Ω_G). This means that

$$\{f \circ \pi_1, g \circ \pi_1\}_{T^*G\times\mathbb{R}} = \{f, g\}_{T^*G} \circ \pi_1 \tag{3.27}$$

for $f, g \in C^\infty(T^*G, \mathbb{R})$, $\{,\}_{T^*G\times\mathbb{R}}$ (respectively, $\{,\}_{T^*G}$) being the Jacobi bracket (respectively, Poisson bracket) associated with the contact 1-form η_G (respectively, the symplectic 2-form Ω_G).

Now, suppose that $\{,\}_0$ is the Jacobi bracket associated with the Jacobi structure (Λ_0, E_0). From (3.6), it follows that

$$\tilde{\alpha}_\sigma^*\{\tilde{f}, \tilde{g}\}_0 = \mathrm{e}^{-\bar{\sigma}}\{\mathrm{e}^{\bar{\sigma}}\tilde{\alpha}_\sigma^*\tilde{f},\ \mathrm{e}^{\bar{\sigma}}\tilde{\alpha}_\sigma^*\tilde{g}\}_{T^*G\times\mathbb{R}} \tag{3.28}$$

for $\tilde{f}, \tilde{g} \in C^\infty(T^*G \times \mathbb{R}, \mathbb{R})$. Thus, if $X, Y \in \Gamma(AG)$ and $\tilde{X}, \tilde{Y}$ are the corresponding linear functions on A^*G, then (see (3.15), (3.27) and (3.28))

$$\begin{aligned}\{\tilde{X}, \tilde{Y}\}_0(\tilde{\alpha}_\sigma(\omega_g, \gamma)) &= (\mathrm{e}^{-\bar{\sigma}}\{\tilde{\alpha}^*(\tilde{X}) \circ \pi_1, \tilde{\alpha}^*(\tilde{Y}) \circ \pi_1\}_{T^*G\times\mathbb{R}})(\omega_g, \gamma) \\ &= \mathrm{e}^{-\sigma(g)}\{\tilde{\alpha}^*(\tilde{X}), \tilde{\alpha}^*(\tilde{Y})\}_{T^*G}(\omega_g) \end{aligned} \tag{3.29}$$

for $(\omega_g, \gamma) \in T_g^*G \times \mathbb{R}$. On the other hand, using the results in [2], we have that

$$(\pi_G)_*^{\nu_h}(X^{\Omega_G}_{\tilde{\alpha}^*(\tilde{X})}(\nu_h)) = \overleftarrow{X}(h), \qquad (\pi_G)_*^{\nu_h}(X^{\Omega_G}_{\tilde{\alpha}^*(\tilde{Y})}(\nu_h)) = \overleftarrow{Y}(h) \tag{3.30}$$

for $h \in G$ and $\nu_h \in T_h^*G$, where $X^{\Omega_G}_{\tilde{\alpha}^*(\tilde{X})}$ (respectively, $X^{\Omega_G}_{\tilde{\alpha}^*(\tilde{Y})}$) is the Hamiltonian vector field of the function $\tilde{\alpha}^*(\tilde{X})$ (respectively, $\tilde{\alpha}^*(\tilde{Y})$) with respect to the symplectic structure Ω_G. Therefore, $\mathcal{L}_{X^{\Omega_G}_{\tilde{\alpha}^*(\tilde{X})}}\lambda_G = \mathcal{L}_{X^{\Omega_G}_{\tilde{\alpha}^*(\tilde{Y})}}\lambda_G = 0$ and from (3.29) and (3.30), we conclude that

$$\begin{aligned}\{\tilde{X}, \tilde{Y}\}_0(\tilde{\alpha}_\sigma(\omega_g, \gamma)) &= \mathrm{e}^{-\sigma(g)}\lambda_G(\omega_g)[X^{\Omega_G}_{\tilde{\alpha}^*(\tilde{X})}, X^{\Omega_G}_{\tilde{\alpha}^*(\tilde{Y})}](\omega_g) = \mathrm{e}^{-\sigma(g)}\omega_g(\overleftarrow{[\![X, Y]\!]}(g)) \\ &= \tilde{\alpha}_\sigma(\omega_g, \gamma)([\![X, Y]\!](\alpha(g))),\end{aligned}$$

$[\![\,,\,]\!]$ being the Lie bracket on AG. Consequently:

$$\{\tilde{X}, \tilde{Y}\}_0 = \widetilde{[\![X, Y]\!]}. \tag{3.31}$$

Next, we will show that

$$\{\tilde{X}, 1\}_0 = \phi_0(X) \circ \tau_{A^*G}, \tag{3.32}$$

where $\tau_{A^*G} : A^*G \to M$ is the bundle projection. Using (3.15), (3.27) and (3.28), it follows that

$$\begin{aligned}\{\tilde{X}, 1\}_0(\tilde{\alpha}_\sigma(\omega_g, \gamma)) &= (\mathrm{e}^{-\bar{\sigma}}\{\tilde{\alpha}^*(\tilde{X}) \circ \pi_1,\ \mathrm{e}^{\sigma\circ\pi_G} \circ \pi_1\}_{T^*G\times\mathbb{R}})(\omega_g, \gamma) \\ &= \mathrm{e}^{-\sigma(g)}\{\tilde{\alpha}^*(\tilde{X}),\ \mathrm{e}^{\sigma\circ\pi_G}\}_{T^*G}(\omega_g) \\ &= (\pi_G)_*^{\omega_g}(X^{\Omega_G}_{\tilde{\alpha}^*(\tilde{X})}(\omega_g))(\sigma).\end{aligned}$$

Thus, from (3.26) and (3.30), we obtain that

$$\{\tilde{X}, 1\}_0(\tilde{\alpha}_\sigma(\omega_g, \gamma)) = (\phi_0(X) \circ \tau_{A^*G})(\tilde{\alpha}_\sigma(\omega_g, \gamma)).$$

This implies that (3.32) holds.

Finally, using (3.31) and (3.32), we deduce (3.25). □

4. Jacobi groupoids

4.1. Jacobi groupoids: definition and examples

Motivated by the results obtained in Section 3 about contact groupoids, we introduce the following definition.

Definition 4.1. Let $G \rightrightarrows M$ be a Lie groupoid, (Λ, E) be a Jacobi structure on G and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid if the homomorphism $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ given by

$$\#_{(\Lambda,E)}(\omega_g, \gamma) = (\#_\Lambda(\omega_g) + \gamma E_g, -\omega_g(E_g))$$

is a morphism of Lie groupoids over some map $\varphi_0 : A^*G \to TM \times \mathbb{R}$, where the structural functions of the Lie groupoid structure on $T^*G \times \mathbb{R} \rightrightarrows A^*G$ (respectively, $TG \times \mathbb{R} \rightrightarrows TM \times \mathbb{R}$) are given by (3.15) (respectively, (3.8)).

Remark 4.2. Since $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a morphism of Lie groupoids, we deduce that

$$\varphi_0 = (\alpha^{\mathrm{T}})_\sigma \circ \#_{(\Lambda,E)} \circ \tilde{\epsilon}_\sigma = (\beta^{\mathrm{T}})_\sigma \circ \#_{(\Lambda,E)} \circ \tilde{\epsilon}_\sigma.$$

Thus, if $\omega_x \in A^*_xG$, it follows that

$$\varphi_0(\omega_x) = (\alpha_*^{\tilde{x}}(\#_\Lambda(\tilde{\epsilon}(\omega_x))), -\omega_x(E(\tilde{x}) - \epsilon_*^x(\beta_*^{\tilde{x}}(E(\tilde{x}))))). \tag{4.1}$$

Example 4.3.

1. *Poisson groupoids*
   If $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid with $E = 0$ and $\sigma = 0$, then we recover the definition of a Poisson groupoid (see [30,31] and Remark 3.8).
2. *Contact groupoids*
   Let $(G \rightrightarrows M, \eta, \sigma)$ be a contact groupoid. If (Λ, E) is the Jacobi structure associated with the contact 1-form η then, using the results in Section 3, we have that $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid.
3. *Jacobi–Lie groups*
   In [18], we proved that generalized Lie bialgebras (i.e. generalized Lie bialgebroids over a single point) may be considered as the infinitesimal invariants of a particular class of Lie groups. These Lie groups can be defined as follows. Let G be a Lie group with identity element $\mathfrak{e}$, $\sigma : G \to \mathbb{R}$ be a multiplicative function and (Λ, E) be a Jacobi structure on G such that:

   (i) Λ is σ-multiplicative, i.e. $\Lambda(gh) = (R_h)_*^g(\Lambda(g)) + \mathrm{e}^{-\sigma(g)}(L_g)_*^h(\Lambda(h))$ for $g, h \in G$.
   (ii) E is a right-invariant vector field, $E(\mathfrak{e}) = -X_0$.
   (iii) $\#_\Lambda(\delta\sigma) = \overrightarrow{X}_0 - \mathrm{e}^{-\sigma}\overleftarrow{X}_0$.

   Condition (i) implies that $\Lambda(\mathfrak{e}) = 0$ and conditions (ii) and (iii) imply that $E(\sigma) = 0$. Thus, using again (ii) and (iii), we deduce that

   $$(\alpha^{\mathrm{T}})_\sigma \circ \#_{(\Lambda,E)} = \varphi_0 \circ \tilde{\alpha}_\sigma, \qquad (\beta^{\mathrm{T}})_\sigma \circ \#_{(\Lambda,E)} = \varphi_0 \circ \tilde{\beta}_\sigma.$$

   In addition, from conditions (ii) and (iii), we have that

   $$(L_g)_*^h E_h = \mathrm{e}^{\sigma(g)}(E_{gh} + (R_h)_*^g(\#_\Lambda(\delta\sigma)_g)) \tag{4.2}$$

   for $g, h \in G$.

Now, suppose that $(\omega_g, \gamma) \in T_g^*G \times \mathbb{R}$ and $(\nu_h, \zeta) \in T_h^*G \times \mathbb{R}$ satisfy the condition $\tilde{\alpha}_\sigma(\omega_g, \gamma) = \tilde{\beta}_\sigma(\nu_h, \zeta)$. Then

$$\tilde{\alpha}(\omega_g + \zeta\, \mathrm{e}^{\sigma(g)}(\delta\sigma)_g) - \tilde{\beta}(\mathrm{e}^{\sigma(g)}\nu_h). \tag{4.3}$$

Thus, using (2.12), (3.8) and (4.2) and the fact that E is a right-invariant vector field, we deduce that

$$\begin{aligned}\#_{(\Lambda,E)}(\omega_g, \gamma) \oplus_{TG\times\mathbb{R}} \#_{(\Lambda,E)}(\nu_h, \zeta) &\\ = ((L_g)_*^h(\#_\Lambda(\nu_h)) + (R_h)_*^g(\#_\Lambda(\omega_g + \zeta\, \mathrm{e}^{\sigma(g)}(\delta\sigma)_g)) &\\ + (\gamma + \mathrm{e}^{\sigma(g)}\zeta)E_{gh}, -\omega_g(E_g)). &\end{aligned}$$

On the other hand, $E_{gh} = E_g \oplus_{TG} 0_h$, and therefore, from (2.14) and (3.15), it follows that

$$\begin{aligned}\#_{(\Lambda,E)}((\omega_g, \gamma) \oplus_{T^*G\times\mathbb{R}} (\nu_h, \zeta)) &\\ = (\tfrac{1}{2}\#_\Lambda\{((R_{h^{-1}})_*^{gh})^*(\omega_g + \zeta\, \mathrm{e}^{\sigma(g)}(\delta\sigma)_g) + ((L_{g^{-1}})_*^{gh})^*(\mathrm{e}^{\sigma(g)}\nu_h)\} &\\ + (\gamma + \mathrm{e}^{\sigma(g)}\zeta)E_{gh}, -\omega_g(E_g)). &\end{aligned}$$

Consequently, using (2.13) and (4.3) and the fact that Λ is σ-multiplicative, we conclude that

$$\#_{(\Lambda,E)}(\omega_g, \gamma) \oplus_{TG\times\mathbb{R}} \#_{(\Lambda,E)}(\nu_h, \zeta) = \#_{(\Lambda,E)}((\omega_g, \gamma) \oplus_{T^*G\times\mathbb{R}} (\nu_h, \zeta)).$$

Thus, we have proved that $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid.

4. *An abelian Jacobi groupoid*

   Let $(L, [\![\,,\,]\!], \rho)$ be a Lie algebroid over M and Λ_{L^*} be the corresponding linear Poisson structure on the dual bundle L^* (see Section 2.2). We may consider on L^* the Lie groupoid structure for which $\alpha = \beta$ is the vector bundle projection and the partial multiplication is the addition in the fibers. Then, L^* with the Poisson structure Λ_{L^*} is a Poisson groupoid (see [40]).

   Now, suppose that $\omega_0 \in \Gamma(L^*)$ is a 1-cocycle of L and denote by $(\Lambda_{(L^*,\omega_0)}, E_{(L^*,\omega_0)})$ the Jacobi structure on L^* given by (3.23). Note that: (i) The Liouville vector field Δ of L^* and the vertical lift $\omega_0^v \in \mathfrak{X}(L^*)$ of ω_0 to L^* are α-vertical and β-vertical vector fields on L^*, and (ii) ω_0^v is a right-invariant and left-invariant vector field on L^*. Using (i), (ii), (2.11), (3.8), (3.15), (3.23) and the fact that (L^*, Λ_{L^*}) is a Poisson groupoid, we deduce that $(L^* \rightrightarrows M, \Lambda_{(L^*,\omega_0)}, E_{(L^*,\omega_0)}, 0)$ is a Jacobi groupoid.
5. *The banal Jacobi groupoid*

   Let M be a differentiable manifold. The results in Section 2.3 (see Example 2.8) imply that $G = M \times \mathbb{R} \times M$ is a Lie groupoid over M and, moreover, the function $\sigma : G \to \mathbb{R}$ given by $\sigma(x, t, y) = t$ is multiplicative. Thus, we can consider the corresponding Lie groupoids $TG \times \mathbb{R} \rightrightarrows TM \times \mathbb{R}$ and $T^*G \times \mathbb{R} \rightrightarrows A^*G$.

   On the other hand, the map $\Phi : TM \times \mathbb{R} \to AG$ given by

$$\Phi(X_x, \lambda) = \left(0, \lambda \frac{\partial}{\partial t|_0}, X_x\right) \in T_{(x,0,x)}G \quad \text{for} \ \ (X_x\lambda) \in T_xM \times \mathbb{R} \tag{4.4}$$

defines an isomorphism between the Lie algebroids $(TM \times \mathbb{R}, [\,,\,], \pi)$ (see Section 2.2) and AG. Thus, AG may be identified with $TM \times \mathbb{R}$ and, under this identification, the projections and the partial multiplications on $TG \times \mathbb{R}$ and $T^*G \times \mathbb{R}$ are given by

$$\begin{aligned}
&(\alpha^{\mathrm{T}})_\sigma \left(\left(X_x, a\frac{\partial}{\partial t|_t}, Y_y \right), \lambda \right) = (Y_y, a+\lambda),\\
&(\beta^{\mathrm{T}})_\sigma \left(\left(X'_{x'}, a'\frac{\partial}{\partial t|_{t'}}, Y'_{y'} \right), \lambda' \right) = (X'_{x'}, \lambda'),\\
&\left(\left(X_x, a\frac{\partial}{\partial t|_t}, Y_y \right), \lambda \right) \oplus_{TG\times\mathbb{R}} \left(\left(Y_y, a'\frac{\partial}{\partial t|_{t'}}, Y'_{y'} \right), a+\lambda \right)\\
&\quad = \left(\left(X_x, (a+a')\frac{\partial}{\partial t|_{t+t'}}, Y'_{y'} \right), \lambda \right),\\
&\tilde{\alpha}_\sigma((\omega_x, a\delta t|_t, \theta_y), \gamma) = (\mathrm{e}^{-t}\theta_y, \gamma),\\
&\tilde{\beta}_\sigma((\omega'_{x'}, a'\delta t|_{t'}, \theta'_{y'}), \gamma') = (-\omega'_{x'}, a'-\gamma'),\\
&((\omega_x, a\delta t|_t, \theta_y), \gamma) \oplus_{T^*G\times\mathbb{R}} ((-\mathrm{e}^{-t}\theta_y, a'\delta t|_{t'}, \theta'_{y'}), a' - \mathrm{e}^{-t}a)\\
&\quad = ((\omega_x, a'\,\mathrm{e}^{t}\delta t|_{t+t'}, \mathrm{e}^{t}\theta'_{y'}), \gamma - a + \mathrm{e}^{t}a').
\end{aligned}$$

Now, suppose that (Λ, E) is a Jacobi structure on M. Then, it was proved in [15] that the pair (Λ', E') is a Jacobi structure on G, where

$$\begin{aligned}
&\Lambda'(x,t,y) = -\left(\Lambda(x) - \frac{\partial}{\partial t|_t} \wedge E(x) \right) + \mathrm{e}^{-t}\left(\Lambda(y) + \frac{\partial}{\partial t|_t} \wedge E(y) \right),\\
&E'(x,t,y) = -E(x).
\end{aligned} \tag{4.5}$$

Furthermore, it is easy to prove that the map $\varphi_0 : A^*G \cong T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ given by (4.1) is just the homomorphism $\#_{(\Lambda,E)} : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$. Using the above facts, we conclude that $(G \rightrightarrows M, \Lambda', E', \sigma)$ is a Jacobi groupoid.

4.2. Some basic properties of Jacobi groupoids

In this section, we will show some basic properties of Jacobi groupoids.

Proposition 4.4. *Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid. Then*:

(i) *$M \cong \epsilon(M)$ is a coisotropic submanifold in G.*
(ii) *E is a right-invariant vector field on G and $E(\sigma) = 0$. Moreover, if $X_0 \in \Gamma(AG)$ is the section of the Lie algebroid AG of G satisfying $E = -\overrightarrow{X}_0$, we have that*

$$\#_\Lambda(\delta\sigma) = \overrightarrow{X}_0 - \mathrm{e}^{-\sigma}\overleftarrow{X}_0. \tag{4.6}$$

(iii) *If $\tilde{\alpha}$, $\tilde{\beta}$* and *$\tilde{\epsilon}$ are the projections and the inclusion of the Lie groupoid $T^*G \rightrightarrows A^*G$, then*

$$\mathrm{e}^{-\sigma}\#_\Lambda \circ \tilde{\epsilon} \circ \tilde{\alpha} = \epsilon^{\mathrm{T}} \circ \alpha^{\mathrm{T}} \circ \#_\Lambda, \qquad \#_\Lambda \circ \tilde{\epsilon} \circ \tilde{\beta} = \epsilon^{\mathrm{T}} \circ \beta^{\mathrm{T}} \circ \#_\Lambda.$$

(iv) *If g and h are elements of G such that $\alpha(g) = \beta(h) = x$ and $\mathcal{X}$ and $\mathcal{Y}$ are (local) bisections through the points g and h, $\mathcal{X}(x) = g$ and $\mathcal{Y}(x) = h$, then*

$$\Lambda(gh) = (R_{\mathcal{Y}})_*^g(\Lambda(g)) + \mathrm{e}^{-\sigma(g)}(L_{\mathcal{X}})_*^h(\Lambda(h)) - \mathrm{e}^{-\sigma(g)}(L_{\mathcal{X}} \circ R_{\mathcal{Y}})_*^{\tilde{x}}(\Lambda(\tilde{x})). \tag{4.7}$$

Proof. If x is a point of M then, using (2.13), we obtain that the map

$$\tilde{\epsilon}_{|A_x^*G} : A_x^*G \to T_{\tilde{x}}^*G$$

is a linear isomorphism between the vector spaces A_x^*G and the annihilator of the subspace$T_{\tilde{x}}\epsilon(M)$, i.e. $(T_{\tilde{x}}\epsilon(M))^\circ$. Thus, from (2.10), (2.13), (3.8), (3.15) and (4.1) and since $(\epsilon^{\mathrm{T}})_\sigma \circ \varphi_0 = \#_{(\Lambda,E)} \circ \tilde{\epsilon}_\sigma$, it follows that $M \cong \epsilon(M)$ is a coisotropic submanifold in G with respect to Λ. This proves (i).

On the other hand, using (2.10), (2.13), (3.8), (3.15) and (4.1) and the relations:

$$(\alpha^{\mathrm{T}})_\sigma \circ \#_{(\Lambda,E)} = \varphi_0 \circ \tilde{\alpha}_\sigma, \qquad (\beta^{\mathrm{T}})_\sigma \circ \#_{(\Lambda,E)} = \varphi_0 \circ \tilde{\beta}_\sigma,$$

we deduce (ii) and (iii).

Finally, we will prove (iv). Using the multiplicative function σ, one may introduce the Lie groupoid structure in T^*G over A^*G with structural functions $\tilde{\alpha}_\sigma^*$, $\tilde{\beta}_\sigma^*$, $\oplus_{T^*G}^\sigma$, and $\tilde{\epsilon}_\sigma^*$ and $\tilde{\iota}_\sigma^*$ given by

$$\begin{aligned}
&\tilde{\alpha}_\sigma^*(\omega_g) = \mathrm{e}^{-\sigma(g)}\tilde{\alpha}(\omega_g), \qquad \tilde{\beta}_\sigma^*(\nu_h) = \tilde{\beta}(\nu_h) \quad \text{for } \omega_g \in T_g^*G \quad \text{and} \quad \nu_h \in T_h^*G,\\
&(\omega_g \oplus_{T^*G}^\sigma \nu_h) = \omega_g \oplus_{T^*G} (\mathrm{e}^{\sigma(g)}\nu_h), \qquad \tilde{\epsilon}_\sigma^*(\omega_x) = \tilde{\epsilon}(\omega_x) \quad \text{for } \omega_x \in A_x^*G,\\
&\tilde{\iota}_\sigma^*(\omega_g) = \mathrm{e}^{-\sigma(g)}\tilde{\iota}(\omega_g) \quad \text{for } \omega_g \in T_g^*G.
\end{aligned} \tag{4.8}$$

In fact, if we consider on $T^*G \times \mathbb{R}$ the Lie groupoid structure over A^*G introduced in Section 3, then the canonical inclusion

$$T^*G \to T^*G \times \mathbb{R}, \qquad \omega_g \in T_g^*G \mapsto (\omega_g, 0) \in T_g^*G \times \mathbb{R}$$

is a Lie groupoid monomorphism over the identity of A^*G.

Since the map $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a Lie groupoid homomorphism, we have that (see (3.8), (3.15) and (4.8))

$$\#_\Lambda(\omega_g \oplus_{T^*G} \nu_h) = \#_\Lambda(\omega_g) \oplus_{TG} \#_\Lambda(\mathrm{e}^{-\sigma(g)}\nu_h)$$

for $\omega_g \in T_g^*G$ and $\nu_h \in T_h^*G$ satisfying $\tilde{\alpha}(\omega_g) = \tilde{\beta}(\nu_h)$. Thus, if Π is the 2-vector on $G \times G \times G$ defined by $\Pi(g,h,k) = \mathrm{e}^{\sigma(g)}\Lambda(g) + \Lambda(h) - \mathrm{e}^{\sigma(g)}\Lambda(k)$, it follows that the graph of the multiplication in G, $\{(g,h,gh) \in G \times G \times G/\alpha(g) = \beta(h)\}$, is a coisotropic submanifold of $G \times G \times G$ with respect to Π.

Now, denote by Ω the affinoid diagram corresponding to the Lie groupoid G, i.e. see [41]:

$$\Omega = \{(k,g,h,r) \in G \times G \times G \times G/\alpha(h) = \alpha(k), \beta(k) = \beta(g), r = hk^{-1}g\}.$$

Then, following the proof of Theorem 4.5 in [41], we obtain that Ω is a coisotropic submanifold of $G \times G \times G \times G$ with respect to the 2-vector $\tilde{\Pi}$ given by

$$\tilde{\Pi}(k, g, h, r) = \mathrm{e}^{\sigma(k)}\Lambda(k) - \mathrm{e}^{\sigma(k)}\Lambda(g) - \mathrm{e}^{\sigma(h)}\Lambda(h) + \mathrm{e}^{\sigma(h)}\Lambda(r).$$

On the other hand, if g and h are elements of G satisfying $\alpha(g) = \beta(h) = x$, we have that $(gh, g, h, \tilde{x})$ is an element of Ω. In addition for any $\xi \in T^*_{gh}G$ and $\mathcal{X}, \mathcal{Y}$ (local) bisections of G through the points g and $h(\mathcal{X}(x) = g$ and $\mathcal{Y}(x) = h)$, it follows from Lemma 2.6 in [42] that

$$(-\xi, ((R_{\mathcal{Y}})^g_*)^*(\xi), ((L_{\mathcal{X}})^h_*)^*(\xi), -((R_{\mathcal{Y}} \circ L_{\mathcal{X}})^{\tilde{x}}_*)^*(\xi))$$

is a conormal vector to Ω at $(gh, g, h, \tilde{x})$, i.e. it is an element of $(T_{(gh,g,h,\tilde{x})}\Omega)^\circ$. Therefore, if $\xi, \eta \in T^*_{gh}G$ we deduce that

$$\begin{aligned}(&\mathrm{e}^{\sigma(gh)}\Lambda(gh) - \mathrm{e}^{\sigma(h)}(L_{\mathcal{X}})^h_*(\Lambda(h)) - \mathrm{e}^{\sigma(gh)}(R_{\mathcal{Y}})^g_*(\Lambda(g))\\ &+\mathrm{e}^{\sigma(h)}(R_{\mathcal{Y}} \circ L_{\mathcal{X}})^{\tilde{x}}_*(\Lambda(\tilde{x})))(\xi, \eta) = 0.\end{aligned}$$

This implies that (4.7) holds. □

Motivated by the above result, we introduce the following definition.

Definition 4.5. Let $G \rightrightarrows M$ be a Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function. A multivector field P on G is σ-affine if for any $g, h \in G$ such that $\alpha(g) = \beta(h) = x$ and any (local) bisections $\mathcal{X}, \mathcal{Y}$ through the points g, h, $\mathcal{X}(x) = g$ and $\mathcal{Y}(x) = h$, we have

$$P(gh) = (R_{\mathcal{Y}})^g_*(P(g)) + \mathrm{e}^{-\sigma(g)}(L_{\mathcal{X}})^h_*(P(h)) - \mathrm{e}^{-\sigma(g)}(L_{\mathcal{X}} \circ R_{\mathcal{Y}})^{\tilde{x}}_*(P(\tilde{x})). \tag{4.9}$$

It is clear that if P is a σ-affine multivector and σ identically vanishes, then P is affine (see [31,42]). On the other hand, if G is a Lie group with identity element $\mathfrak{e}$ and P is a σ-affine multivector field on G such that $P(\mathfrak{e}) = 0$, then P is a σ-multiplicative multivector field in the sense of [18].

The following proposition gives a very useful characterization of σ-affine multivector fields (see [18] for the corresponding result for the case of Lie groups).

Proposition 4.6. *Let $G \rightrightarrows M$ be an α-connected Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function on G. For a multivector field P on G, the following statements are equivalent*:

(i) *P is σ-affine,*
(ii) *for any left-invariant vector field $\overleftarrow{X}$, the Lie derivative $\mathrm{e}^\sigma \mathcal{L}_{\overleftarrow{X}} P$ is left-invariant.*

Proof. The result follows using the fact that σ is multiplicative and proceeding as in the proof of Theorem 2.2 in [31]. □

5. Jacobi groupoids and generalized Lie bialgebroids

The aim of this section is to show the relation between Jacobi groupoids and generalized Lie bialgebroids.

5.1. Coisotropic submanifolds of a Jacobi manifold, Lie algebroids and 1-cocycles

In this section, we will prove that if S is a coisotropic submanifold of a Jacobi manifold M, then there exists a Lie algebroid structure on the conormal bundle to S and, in addition, we can define a distinguished 1-cocycle for this Lie algebroid structure. For this purpose, we will need the following result.

Lemma 5.1. *Let (M, Λ, E) be a Jacobi manifold and $([\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ be the Lie algebroid structure on $T^*M \times \mathbb{R}$. Suppose that S is a coisotropic submanifold of M and that $\bar{\jmath}^* : \Omega^1(M) \times C^\infty(M,\mathbb{R}) \to \Omega^1(S) \times C^\infty(S,\mathbb{R})$ is the map defined by $\bar{\jmath}^*(\omega, f) = (\jmath^* \omega, \jmath^* f)$, $\jmath: S \to M$ being the canonical inclusion. Then:*

- (i) *Ker $\bar{\jmath}^*$ is a Lie subalgebra of the Lie algebra $(\Omega^1(M) \times C^\infty(M,\mathbb{R}), [\![,]\!]_{(\Lambda,E)})$.*
- (ii) *The subspace of $\Omega^1(M) \times C^\infty(M,\mathbb{R})$ defined by $\{(\omega, f) \in \Omega^1(M) \times C^\infty(M,\mathbb{R}) / \omega|_S = 0, \jmath^* f = 0\}$ is an ideal in Ker$\bar{\jmath}^*$.*

Proof.

(i) If $(\omega, f), (\nu, g) \in \Omega^1(M) \times C^\infty(M,\mathbb{R})$ satisfy

$$\bar{\jmath}^*(\omega, f) = 0, \qquad \bar{\jmath}^*(\nu, g) = 0,$$

it follows from (2.5) that

$$\bar{\jmath}^*[\![(\omega, f), (\nu, g)]\!]_{(\Lambda,E)} = (\jmath^* (i(\#_\Lambda(\omega))\delta\nu - i(\#_\Lambda(\nu))\delta\omega - \delta(\omega(\#_\Lambda(\nu)))), \jmath^* (\omega(\#_\Lambda(\nu)) + \#_\Lambda(\omega)(g) - \#_\Lambda(\nu)(f))). \quad (5.1)$$

Now, since $\jmath^* \omega = 0$, $\jmath^* \nu = 0$ and S is a coisotropic submanifold, it follows that the restriction to S of the vector fields $\#_\Lambda(\omega)$ and $\#_\Lambda(\nu)$ is tangent to S. Thus, from (5.1), we deduce that

$$\bar{\jmath}^*[\![(\omega, f), (\nu, g)]\!]_{(\Lambda,E)} = 0.$$

(ii) If ω' and ν' are 1-form on M, we will denote by $[\![\omega', \nu']\!]_\Lambda$ the 1-form on M given by

$$[\![\omega', \nu']\!]_\Lambda = i(\#_\Lambda(\omega'))\delta\nu' - i(\#_\Lambda(\nu'))\delta\omega' - \delta(\omega'(\#_\Lambda(\nu'))).$$

Note that

$$[\![\omega', f\nu']\!]_\Lambda = f[\![\omega', \nu']\!]_\Lambda + \#_\Lambda(\omega')(f)\nu' \quad \text{for } f \in C^\infty(M,\mathbb{R}). \quad (5.2)$$

Next, suppose that $(\omega, f), (\nu, g) \in \Omega^1(M) \times C^\infty(M,\mathbb{R})$ satisfy the following conditions:

$$\omega|_S = 0, \qquad \jmath^* f = 0, \qquad \bar{\jmath}^*(\nu, g) = 0.$$

Then, proceeding as in the proof of (i), we have that

$$[\![(\omega, f), (\nu, g)]\!]_{(\Lambda,E)|_S} = ([\![\omega, \nu]\!]_{\Lambda|_S}, 0).$$

Thus, if x is a point of S, we must prove that $[\![\omega, \nu]\!]_\Lambda(x) = 0$. For this purpose, we consider a coordinate neighborhood (U, φ) of M with coordinates $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ such that

$$\varphi(U \cap S) = \{(x_1, \dots, x_m) \in \varphi(U)/x_{n+1} = \cdots = x_m = 0\}.$$

Here, n (respectively, m) is the dimension of S (respectively, M). Then, on U

$$\omega = \sum_{i=1}^{m} \omega^i \delta x_i, \qquad \nu = \sum_{j=1}^{n} \nu^j \delta x_j + \sum_{k=n+1}^{m} \bar{\nu}^k \delta x_k \tag{5.3}$$

with

$$\jmath^* \omega^i = 0, \qquad \jmath^* \nu^j = 0 \tag{5.4}$$

for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

Note that, since S is a coisotropic submanifold of M, it follows that

$$\#_\Lambda(\delta x_k)|_S(\omega^i) = 0 \quad \text{for all } i \in \{1, \dots, m\} \quad \text{and} \quad k \in \{n+1, \dots, m\}. \tag{5.5}$$

Therefore, using (5.2)–(5.5), we conclude that $[\![\omega, \nu]\!]_\Lambda(x) = 0$. □

Now, we will show the main result of the section.

Proposition 5.2. *Let (M, Λ, E) be a Jacobi manifold and $([\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ be the Lie algebroid structure on $T^*M \times \mathbb{R}$. Suppose that S is a coisotropic submanifold of M. Then*:

(i) *The conormal bundle to S, $N(S) = (TS)^\circ \to S$, admits a Lie algebroid structure $([\![,]\!]_S, \rho_S)$ defined by*

$$[\![\omega, \nu]\!]_S(x) = (\pi_1 [\![(\tilde{\omega}, 0), (\tilde{\nu}, 0)]\!]_{(\Lambda,E)})(x), \qquad \rho_S(\omega)(x) = \#_\Lambda(\omega_x) \tag{5.6}$$

for all $x \in S$, where $\pi_1 : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \Omega^1(M)$ is the projection onto the first factor and $\tilde{\omega}$ and $\tilde{\nu}$ are arbitrary extensions to M of ω and ν, respectively.

(ii) *The section E_S of the vector bundle $N(S)^* \to S$ characterized by*

$$\omega(E_S(x)) = -\omega(E(x)) \tag{5.7}$$

for all $\omega \in N_xS = (T_xS)^\circ$ and $x \in S$, is a 1-cocycle of the Lie algebroid $(N(S), [\![,]\!]_S, \rho_S)$.

Proof. (i) follows from Lemma 5.1 and (ii) follows using (5.7) and the fact that $(-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ is a 1-cocycle of the Lie algebroid $(T^*M \times \mathbb{R}, [\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$. □

Remark 5.3. If the Jacobi manifold M is Poisson (i.e. $E = 0$), then the 1-cocycle E_S identically vanishes and $([\![,]\!]_S, \rho_S)$ is just the Lie algebroid structure obtained by Weinstein in [40].

5.2. The generalized Lie bialgebroid of a Jacobi groupoid

In this section, we will show that generalized Lie bialgebroids are the infinitesimal invariants for Jacobi groupoids.

Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid and AG be the Lie algebroid of G. Then, E is a right-invariant vector field and, thus, there exists a section X_0 of AG such that $E = -\overrightarrow{X}_0$ (see Proposition 4.4). Moreover, the conormal bundle to M, as a submanifold of G, may be identified with A^*G.

Now, we consider the section ϕ_0 of A^*G given by

$$\phi_0(X_x) = X_x(\sigma) \quad \text{for} \quad X_x \in A_xG \quad \text{and} \quad x \in M. \tag{5.8}$$

Since σ is a Lie groupoid 1-cocycle, it follows that ϕ_0 is a 1-cocycle of the Lie algebroid AG (see [42]).

On the other hand, using that $M \cong \epsilon(M)$ is a coisotropic submanifold of G, we deduce that there exists a Lie algebroid structure $([\![\,,\,]\!]_*, \rho_*)$ on A^*G and, furthermore, the vector field E induces a 1-cocycle $E_M \in \Gamma(AG)$ of A^*G (see Proposition 5.2). In fact, from Proposition 5.2, we have that $E_M = X_0$ and

$$\begin{aligned} &[\![\omega, \nu]\!]_*(x) = \pi_1[\![(\widetilde{\tilde{\epsilon} \circ \omega}, 0), (\widetilde{\tilde{\epsilon} \circ \omega}, 0)]\!]_{(\Lambda,E)}(\tilde{x}),\\ &\rho_*(\omega)(x) = \alpha_*^{\tilde{x}}(\#_\Lambda(\tilde{\epsilon}(\omega_x))) \end{aligned} \tag{5.9}$$

for $\omega, \nu \in \Gamma(A^*G)$ and $x \in M$, where $\tilde{\epsilon}$ is the inclusion in the Lie groupoid $T^*G \rightrightarrows A^*G$ and $\widetilde{\tilde{\epsilon} \circ \omega}$ and $\widetilde{\tilde{\nu} \circ \omega}$ are arbitrary extensions to G of $\tilde{\epsilon} \circ \omega$ and $\tilde{\epsilon} \circ \nu$, respectively.

Note that, from (4.1) and (5.9), we have that $\varphi_0 = (\rho_*, X_0)$, where

$$(\rho_*, X_0)(\omega_x) = (\rho_*(\omega_x), \omega_x(X_0(x))) \tag{5.10}$$

for $\omega_x \in A_x^*G$. In addition, we will prove the following result.

Theorem 5.4. *Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid. Then $((AG, \phi_0), (A^*G, X_0))$ is a generalized Lie bialgebroid.*

Proof. Denote by d_{*X_0} the X_0-differential of the Lie algebroid $(A^*G, [\![\,,\,]\!]_*, \rho_*)$. We will show that

$$\mathrm{e}^\sigma \mathcal{L}_{\overleftarrow{X}} \Lambda = -\overleftarrow{\mathrm{d}_{*X_0} X} \tag{5.11}$$

for $X \in \Gamma(AG)$. Suppose that ω_1, ω_2 are any sections of A^*G. Let $\widetilde{\tilde{\epsilon} \circ \omega_1}, \widetilde{\tilde{\epsilon} \circ \omega_2}$ be any of their extensions to 1-forms on G. Then, using (2.3), (2.5), (2.15) and (5.9) and the fact that $\sigma|_{\epsilon(M)} \equiv 0$, we have that

$$\begin{aligned} &(\mathrm{e}^\sigma \mathcal{L}_{\overleftarrow{X}} \Lambda)|_{\epsilon(M)}(\omega_1, \omega_2)\\ &\quad = ((\mathcal{L}_{\#_\Lambda(\widetilde{\tilde{\epsilon}\circ\omega_1})} \widetilde{\tilde{\epsilon} \circ \omega_2} - \mathcal{L}_{\#_\Lambda(\widetilde{\tilde{\epsilon}\circ\omega_2})} \widetilde{\tilde{\epsilon} \circ \omega_1} - \Lambda(\widetilde{\tilde{\epsilon} \circ \omega_1}, \widetilde{\tilde{\epsilon} \circ \omega_2}))(\overleftarrow{X})\\ &\qquad + \#_\Lambda(\widetilde{\tilde{\epsilon} \circ \omega_2})(\widetilde{\tilde{\epsilon} \circ \omega_1}(\overleftarrow{X})) - \#_\Lambda(\widetilde{\tilde{\epsilon} \circ \omega_1})(\widetilde{\tilde{\epsilon} \circ \omega_2}(\overleftarrow{X})))|_{\epsilon(M)}\\ &\quad = [\![\omega_1, \omega_2]\!]_*(X) + \rho_*(\omega_2)(\omega_1(X)) - \rho_*(\omega_1)(\omega_2(X)) - (X_0 \wedge X)(\omega_1, \omega_2)\\ &\quad = -(\mathrm{d}_{*X_0} X)(\omega_1, \omega_2). \end{aligned}$$

Thus, since $-\overleftarrow{\mathrm{d}_{*X_0}X}$ and $\mathrm{e}^{\sigma}\mathcal{L}_{\overleftarrow{X}}\Lambda$ are left-invariant 2-vectors (see Proposition 4.6) and their evaluation coincides on the conormal bundle A^*G, we deduce (5.11).

Using (2.8), (5.8) and (5.11), we obtain that

$$\begin{aligned}\overleftarrow{\mathrm{d}_{*X_0}[\![X,Y]\!]} &= -\mathrm{e}^{\sigma}\mathcal{L}_{[\overleftarrow{X},\overleftarrow{Y}]}\Lambda = \mathcal{L}_{\overleftarrow{Y}}(\mathrm{e}^{\sigma}\mathcal{L}_{\overleftarrow{X}}\Lambda) - \overleftarrow{Y}(\sigma)(\mathrm{e}^{\sigma}\mathcal{L}_{\overleftarrow{X}}\Lambda)\\ &\quad - \mathcal{L}_{\overleftarrow{X}}(\mathrm{e}^{\sigma}\mathcal{L}_{\overleftarrow{Y}}\Lambda) + \overleftarrow{X}(\sigma)(\mathrm{e}^{\sigma}\mathcal{L}_{\overleftarrow{Y}}\Lambda)\\ &= \overleftarrow{[\![X,\mathrm{d}_{*X_0}Y]\!]} - \overleftarrow{\phi_0(X)\mathrm{d}_{*X_0}Y} - \overleftarrow{[\![Y,\mathrm{d}_{*X_0}X]\!]} + \overleftarrow{\phi_0(Y)\mathrm{d}_{*X_0}X} \end{aligned} \tag{5.12}$$

for $X, Y \in \Gamma(AG)$, where $([\![,]\!], \rho)$ is the Lie algebroid structure on AG. Thus, from (2.16) and (5.12), we conclude that

$$\mathrm{d}_{*X_0}[\![X,Y]\!] = [\![X,\mathrm{d}_{*X_0}Y]\!]_{\phi_0} - [\![Y,\mathrm{d}_{*X_0}X]\!]_{\phi_0}$$

for $X, Y \in \Gamma(AG)$.

Now, (5.8), the condition $E(\sigma) = -\overrightarrow{X}_0(\sigma) = 0$ (see Proposition 4.4) and the fact that σ is a multiplicative function imply that $\phi_0(X_0)\circ\alpha = 0$ and, therefore:

$$\phi_0(X_0) = 0. \tag{5.13}$$

Furthermore, if $x \in M$ then, from (4.6), (5.8) and (5.9), we deduce that

$$\epsilon_*^x(\rho_*(\phi_0)(x)) = \#_\Lambda(\delta\sigma)(\tilde{x}) = \overleftarrow{X}_0(\tilde{x}) - \overrightarrow{X}_0(\tilde{x}) = -\epsilon_*^x(\alpha_*^{\tilde{x}}(X_0(x))),$$

i.e. see (2.7):

$$\rho_*(\phi_0)(x) = -\rho(X_0)(x). \tag{5.14}$$

On the other hand, using (5.8), (5.11) and (5.13), it follows that

$$\mathrm{e}^{-\sigma}i(\delta\sigma)(\overleftarrow{\mathrm{d}_*X}) = -i(\delta\sigma)(\mathcal{L}_{\overleftarrow{X}}\Lambda) + \mathrm{e}^{-\sigma}(\phi_0(X)\circ\alpha)\overleftarrow{X}_0.$$

Consequently, using again (5.8), we have that

$$i(\phi_0)(\mathrm{d}_*X) = -i((\delta\sigma)(\mathcal{L}_{\overleftarrow{X}}\Lambda))\circ\epsilon + \phi_0(X)X_0. \tag{5.15}$$

Finally, from (4.6) and (5.8), we deduce that

$$\begin{aligned}0 = [\overleftarrow{X}, \overrightarrow{X}_0] &= i(\delta\sigma)(\mathcal{L}_{\overleftarrow{X}}\Lambda) + \#_\Lambda(\delta(\phi_0(X)\circ\alpha))\\ &\quad - \mathrm{e}^{-\sigma}(\phi_0(X)\circ\alpha)\overleftarrow{X}_0 + \mathrm{e}^{-\sigma}\overleftarrow{[\![X,X_0]\!]},\end{aligned}$$

which implies that (see (2.3), (5.8), (5.9) and (5.15))

$$i(\phi_0)(\mathrm{d}_*X) + \mathrm{d}_*(\phi_0(X)) + [\![X_0,X]\!] = 0. \qquad \square$$

Next, we will describe the generalized Lie bialgebroids associated with some examples of Jacobi groupoids. We remark that two generalized Lie bialgebroids $((A,\phi_0),(A^*,X_0))$ and $((B,\omega_0),(B^*,Y_0))$ over a manifold M are isomorphic if there exists a Lie algebroid isomorphism $\mathcal{I}: A \to B$ such that $\mathcal{I}(X_0) = Y_0$ and, in addition, the adjoint operator $\mathcal{I}^*: B^* \to A^*$ is also a Lie algebroid isomorphism satisfying $\mathcal{I}^*(\omega_0) = \phi_0$.

Example 5.5.

1. *Poisson groupoids*

If (G, Λ, E, σ) is a Jacobi groupoid with $E = 0$ and $\sigma = 0$, i.e. (G, Λ) is a Poisson groupoid, then we have that ϕ_0 and X_0 identically vanish (see (5.8) and Remark 5.3). Therefore, (2.18) and Theorem 5.4 imply a well-known result (see [30]): if (G, Λ) is a Poisson groupoid, then the pair (AG, A^*G) is a Lie bialgebroid.

2. *Contact groupoids*

Let $(G \rightrightarrows M, \eta, \sigma)$ be a contact groupoid and (Λ, E) be the Jacobi structure associated with the contact 1-form η. Then, $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid.

Now, denote by (Λ_0, E_0) the Jacobi structure on M characterized by the conditions (3.6), by X_0 the section of the Lie algebroid AG of G satisfying $E = -\overrightarrow{X}_0$ and by $\mathcal{I} : T^*M \times \mathbb{R} \to AG$ the Lie algebroid isomorphism given by (3.7). If we consider the section $(0, -1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$ of the vector bundle $T^*M \times R \to M$, we have that (see (3.7))

$$\mathcal{I}(0, -1) = X_0. \tag{5.16}$$

Moreover, if $\mathcal{I}^* : A^*G \to TM \times \mathbb{R}$ is the adjoint operator of $\mathcal{I}$, from (3.7), it follows that

$$\mathcal{I}^*(\nu_x) = (-\alpha_*^{\tilde{x}}(\#_\Lambda(\tilde{\epsilon}(\nu_x))), -\nu_x(X_0(x))) \tag{5.17}$$

for $\nu_x \in A_x^*G$, where $\tilde{\epsilon}$ is the inclusion in the Lie groupoid $T^*G \rightrightarrows A^*G$.

Next, denote by $([\,,\,]_-, \pi_-)$ the Lie algebroid structure on the vector bundle $TM \times \mathbb{R} \to M$ defined by

$$[(X, f), (Y, g)] = (-[X, Y], -(X(g) - Y(f))), \quad \pi_-(X, f) = -X$$

for $(X, f) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$.

On the other hand, if on the vector bundle $TG \times \mathbb{R} \to G$ we consider the natural Lie algebroid structure (see Section 2.2), then the map $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a Lie algebroid homomorphism between the Lie algebroids $(T^*G \times \mathbb{R}, [\![\,,\,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ and $TG \times \mathbb{R}$. Using this fact, (5.9) and since $M \cong \epsilon(M)$ is a coisotropic submanifold of G, we deduce that $\mathcal{I}^*$ defines an isomorphism between the Lie algebroids A^*G and $(TM \times \mathbb{R}, [\,,\,]_-, \pi_-)$. In addition, from (5.17) and Proposition 3.3, we obtain that $\mathcal{I}^*(\phi_0) = (-E_0, 0)$.

In conclusion, if on the vector bundle $T^*M \times \mathbb{R} \to M$ (respectively, $TM \times \mathbb{R} \to M$) we consider the Lie algebroid structure $([\![\,,\,]\!]_{(\Lambda_0,E_0)}, \tilde{\#}_{(\Lambda_0,E_0)})$ (respectively, $([\,,\,]_-, \pi_-)$), then the generalized Lie bialgebroids $((AG, \phi_0), (A^*G, X_0))$ and $((T^*M \times \mathbb{R}, (-E_0, 0)), (TM \times \mathbb{R}, (0, -1)))$ are isomorphic. Note that the Jacobi structure on M induced by the generalized Lie bialgebroid $((T^*M \times \mathbb{R}, (-E_0, 0)), (TM \times \mathbb{R}, (0, -1)))$ is just (Λ_0, E_0) (see (2.20)).

3. *Jacobi–Lie groups*

Let G be a Lie group with identity element $\mathfrak{e}$, $\sigma : G \to \mathbb{R}$ be a multiplicative function and (Λ, E) be a Jacobi structure on G such that Λ is σ-multiplicative, E is a right-invariant vector field and

$$\#_\Lambda(\delta\sigma)_{(g)} = -E_g + \mathrm{e}^{-\sigma(g)}(L_g)_*^{\mathfrak{e}}(E(\mathfrak{e})) \quad \text{for all} \quad g \in G.$$

Then, $(G \rightrightarrows \{\mathfrak{e}\}, \Lambda, E, \sigma)$ is a Jacobi groupoid (see Example 4.3).

The Lie algebroid of G is just the Lie algebra $\mathfrak{g}$ of G, i.e. $AG = \mathfrak{g}$ and, from (5.8), it follows that $\phi_0 = (\delta\sigma)(\mathfrak{e})$.

On the other hand, since $\Lambda(\mathfrak{e}) = 0$, one may consider the intrinsic derivative $\delta_{\mathfrak{e}}\Lambda : \mathfrak{g} \to \wedge^2\mathfrak{g}$ of Λ at $\mathfrak{e}$. In fact, using (2.5) and (5.9), we deduce that the Lie bracket $[\,,\,]_*$ on the dual space $A^*G = \mathfrak{g}^*$ of $\mathfrak{g}$ is given by

$$[\omega, \nu]_* = [\omega, \nu]_\Lambda - \omega(E(\mathfrak{e}))\nu + \nu(E(\mathfrak{e}))\omega$$

for $\omega, \nu \in \mathfrak{g}^*$, where $[\,,\,]_\Lambda : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the adjoint map of the intrinsic derivative of Λ at $\mathfrak{e}$. In addition, the 1-cocycle X_0 on $\mathfrak{g}^*$ is $X_0 = -E(\mathfrak{e})$.

Thus, using Theorem 5.4, we conclude that the pair $((\mathfrak{g}, (\delta\sigma)(\mathfrak{e})), (\mathfrak{g}^*, -E(\mathfrak{e})))$ is a generalized Lie bialgebroid over $\{\mathfrak{e}\}$, i.e. a generalized Lie bialgebra. This result was proved in [18] (see Theorem 3.12 in [18]).

4. *An abelian Jacobi groupoid*

Let $(L, [\![\,,\,]\!], \rho)$ be a Lie algebroid over a manifold M and $\omega_0 \in \Gamma(L^*)$ be a 1-cocycle of L. We may consider on L^* the Jacobi structure $(\Lambda_{(L^*,\omega_0)}, E_{(L^*,\omega_0)})$ given by (3.23) and the Lie groupoid structure for which $\alpha = \beta$ is the vector bundle projection $\tau : L^* \to M$ and the partial multiplication is the addition in the fibers. As we know (see Example 4.3), $(L^* \rightrightarrows M, \Lambda_{(L^*,\omega_0)}, E_{(L^*,\omega_0)}, 0)$ is a Jacobi groupoid and we have the corresponding generalized Lie bialgebroid $((A(L^*), \phi_0 = 0), (A^*(L^*), X_0))$.

On the other hand, if $0 : M \to L^*$ is the zero section of L^* and $\mu \in \tau^{-1}(x) = L_x^*$, we will denote by $\mu^v(0(x)) \in T_{0(x)}L_x^*$ the vertical lift of μ to L^* at the point $0(x)$. Then, the map

$$\upsilon : L^* \to A(L^*), \qquad \mu \in L_x^* \mapsto \mu^v(0(x)) \in A_x(L^*)$$

defines an isomorphism between the vector bundles L^* and $A(L^*)$. Moreover, using (3.23) and since $\alpha = \tau$ and the Lie bracket of two left-invariant vector fields on L^* is zero, we conclude that: (i) υ defines an isomorphism between the Lie algebroid L^* (with the trivial Lie algebroid structure) and $A(L^*)$ and (ii) $\upsilon(\omega_0) = X_0$. In addition, if $\upsilon^* : A^*(L^*) \to L$ is the adjoint map of $\upsilon : L^* \to A(L^*)$ then, from (2.5), (2.6), (3.23) and (5.9), we deduce that υ^* induces an isomorphism between the Lie algebroids $A^*(L^*)$ and $(L, [\![\,,\,]\!], \rho)$.

Therefore, we have proved that the generalized Lie bialgebroids $((A(L^*), 0), (A^*(L^*), X_0))$ and $((L^*, 0), (L, \omega_0))$, are isomorphic.

5. *The banal Jacobi groupoid*

Let (M, Λ, E) be a Jacobi manifold and G the product manifold $M \times \mathbb{R} \times M$. Denote by (Λ', E') the Jacobi structure on G given by (4.5) and by $\sigma : G \to \mathbb{R}$ the function defined by $\sigma(x, t, y) = t$. Then, one may consider a Lie groupoid structure in G over M in such a way that $(G \rightrightarrows M, \Lambda', E', \sigma)$ is a Jacobi groupoid (see Example 4.3). Thus, we have the corresponding generalized Lie bialgebroid $((AG, \phi_0), (A^*G, X_0))$. As we know, the map $\Phi : TM \times \mathbb{R} \to AG$ given by (4.4) defines an isomorphism between the Lie algebroids $(TM \times \mathbb{R}, [\,,\,], \pi)$ and AG and, moreover, it follows that $\Phi(-E, 0) = X_0$.

Now, let $\Phi^* : A^*G \to T^*M \times \mathbb{R}$ be the adjoint map of Φ. Then, using (2.5), (4.4), (4.5) and (5.8), we deduce that Φ^* induces an isomorphism between the Lie algebroids A^*G and $(T^*M \times \mathbb{R}, [\![\,,\,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ and, in addition, $\Phi^*(\phi_0) = (0, 1)$.

Therefore, we have proved that the generalized Lie bialgebroids $((AG, \phi_0), (A^*G, X_0))$ and $((TM \times \mathbb{R}, (0, 1)), (T^*M \times \mathbb{R}, (-E, 0)))$ are isomorphic.

To finish this section, we will relate the Jacobi structure on G and the Jacobi structure on M induced by the generalized Lie bialgebroid structure of Theorem 5.4.

Proposition 5.6. *Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid and (Λ_0, E_0) be the Jacobi structure on M induced by the generalized Lie bialgebroid $((AG, \phi_0), (A^*G, X_0))$. Then, the projection β is a Jacobi antimorphism between the Jacobi manifolds (G, Λ, E) and (M, Λ_0, E_0) and the pair $(\alpha, \mathrm{e}^{\sigma})$ is a conformal Jacobi morphism.*

Proof. Denote by $\{,\}$ (respectively, $\{,\}_0$) the Jacobi bracket associated with the Jacobi structure (Λ, E) (respectively, (Λ_0, E_0)). Then, we must prove that

$$\{\beta^* f_1, \beta^* f_2\} = -\beta^*\{f_1, f_2\}_0, \qquad \mathrm{e}^{-\sigma}\{\mathrm{e}^{\sigma}\alpha^* f_1, \mathrm{e}^{\sigma}\alpha^* f_2\} = \alpha^*\{f_1, f_2\}_0$$

for $f_1, f_2 \in C^\infty(M, \mathbb{R})$.

Now, if $(\rho_*, X_0) : \Gamma(A^*G) \to \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ is the map given by (5.10) and $(\rho, \phi_0) : \Gamma(AG) \to \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ is the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by

$$(\rho, \phi_0)(X) = (\rho(X), \phi_0(X)) \tag{5.18}$$

then, from (2.20), (5.10) and (5.18), it follows that

$$\#_{(\Lambda_0,E_0)} = (\rho_*, X_0) \circ (\rho, \phi_0)^*, \tag{5.19}$$

where $(\rho, \phi_0)^* : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \Gamma(A^*G)$ is the adjoint operator of the homomorphism (ρ, ϕ_0).

Using (3.8) and since $(\beta^{\mathrm{T}})_\sigma \circ \#_{(\Lambda,E)} = (\rho_*, X_0) \circ \tilde{\beta}_\sigma$, we have that

$$\begin{aligned}
\{\beta^* f_1, \beta^* f_2\} &= \langle \#_{(\Lambda,E)}(\beta^*\delta f_1, \beta^* f_1), (\beta^*\delta f_2, \beta^* f_2)\rangle \\
&= \langle ((\beta)^{\mathrm{T}}_\sigma \circ \#_{(\Lambda,E)})(\beta^*\delta f_1, \beta^* f_1), (\delta f_2 \circ \beta, \beta^* f_2)\rangle \\
&= \langle ((\rho_*, X_0) \circ \tilde{\beta}_\sigma)(\beta^*\delta f_1, \beta^* f_1), (\delta f_2 \circ \beta, \beta^* f_2)\rangle.
\end{aligned}$$

From (2.7), (2.13), (3.15), (5.8) and (5.18), we deduce that $\tilde{\beta}_\sigma((\beta_*^g)^*(\omega_{\beta(g)}), \lambda) = -(\rho, \phi_0)^*(\omega_{\beta(g)}, \lambda)$ for $(\omega_{\beta(g)}, \lambda) \in T^*_{\beta(g)}M \times \mathbb{R}$. Using this fact and (5.19), we get that

$$\{\beta^* f_1, \beta^* f_2\} = \beta^*\{f_1, f_2\}_M.$$

On the other hand, using (2.7), (2.20) and (3.8), Proposition 4.4 and since $(\alpha^{\mathrm{T}})_\sigma \circ \#_{(\Lambda,E)} = (\rho_*, X_0) \circ \tilde{\alpha}_\sigma$, we obtain that

$$\begin{aligned} \mathrm{e}^{-\sigma}\{e^\sigma \alpha^* f_1, \mathrm{e}^\sigma \alpha^* f_2\} &= \mathrm{e}^{-\sigma}\langle \#_{(\Lambda,E)}(\delta(\mathrm{e}^\sigma \alpha^* f_1), \mathrm{e}^\sigma \alpha^* f_1), (\delta(\mathrm{e}^\sigma \alpha^* f_2), \mathrm{e}^\sigma \alpha^* f_2)\rangle \\ &= \mathrm{e}^\sigma \langle ((\alpha^{\mathrm{T}})_\sigma \circ \#_{(\Lambda,E)})(\alpha^* \delta f_1, \alpha^* f_1), (\delta f_2 \circ \alpha, \alpha^* f_2)\rangle \\ &\quad + \mathrm{e}^\sigma (\alpha^* f_1)\langle \#_{(\Lambda,E)}(\delta\sigma, 0), (\alpha^*(\delta f_2), \alpha^* f_2)\rangle \\ &= \mathrm{e}^\sigma \langle ((\rho_*, X_0) \circ \tilde{\alpha}_\sigma)(\alpha^*(\delta f_1), \alpha^* f_1), (\delta f_2 \circ \alpha, \alpha^* f_2)\rangle \\ &\quad + \alpha^*(f_1 E_0(f_2)). \end{aligned}$$

Now, from (2.7), (2.13), (3.15), (5.8) and (5.18), it follows that $\mathrm{e}^{\sigma(g)}\tilde{\alpha}_\sigma((\alpha_*^g)^*(\omega_{\alpha(g)}), \lambda) = (\rho, \phi_0)^*(\omega_{\alpha(g)}, 0)$ for $(\omega_{\alpha(g)}, \lambda) \in T^*_{\alpha(g)}M \times \mathbb{R}$. Therefore

$$\mathrm{e}^{-\sigma}\{\mathrm{e}^\sigma \alpha^* f_1, \mathrm{e}^\sigma \alpha^* f_2\} = \alpha^*\{f_1, f_2\}_0. \qquad \square$$

5.3. *Integration of generalized Lie bialgebroids*

In this section, we will show a converse of Theorem 5.4, i.e. we will show that one may integrate a generalized Lie bialgebroid and obtain a Jacobi groupoid.

5.3.1. *Jacobi groupoids and Poisson groupoids*

In this first subsection, we will prove that a Poisson groupoid can be obtained from any Jacobi groupoid and we will show the relation between the generalized Lie bialgebroid associated with the Jacobi groupoid and the Lie bialgebroid induced by the Poisson groupoid.

Let $G \rightrightarrows M$ be a Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, using the multiplicative character of σ, we can define a right action of $G \rightrightarrows M$ on the canonical projection $\pi_1 : M \times \mathbb{R} \to M$ as follows:

$$(x, t) \cdot g = (\alpha(g), \sigma(g) + t) \tag{5.20}$$

for $(x, t) \in M \times \mathbb{R}$ and $g \in G$ such that $\beta(g) = x$. Thus, we have the corresponding action groupoid $(M \times \mathbb{R}) * G \rightrightarrows M \times \mathbb{R}$. Moreover, if $(AG, [\![,]\!], \rho)$ is the Lie algebroid of G, the multiplicative function σ induces a 1-cocycle ϕ_0 on AG given by (see (5.8))

$$\phi_0(x)(X_x) = X_x(\sigma) \quad \text{for } x \in M \text{ and } X_x \in A_xG. \tag{5.21}$$

In addition, using the results in Section 2.3 (see (2.9)), we deduce that the $\mathbb{R}$-linear map $* : \Gamma(AG) \to \mathfrak{X}(M \times \mathbb{R})$ defined by

$$X^* = (\rho(X) \circ \pi_1) + (\phi_0(X) \circ \pi_1)\frac{\partial}{\partial t} \tag{5.22}$$

induces an action of AG on the projection $\pi_1 : M \times \mathbb{R} \to M$ and the Lie algebroid of $(M \times \mathbb{R}) * G$ is just the action Lie algebroid $AG \ltimes \pi_1$.

Now, it is easy to prove that $(M \times \mathbb{R}) * G$ may be identified with the product manifold $G \times \mathbb{R}$ and, under this identification, the structural functions of the Lie groupoid are given by

$$\begin{aligned}
&\alpha_\sigma(g,t)=(\alpha(g),\sigma(g)+t)\quad\text{for }\ (g,t)\in G\times\mathbb{R},\\
&\beta_\sigma(h,s)=(\beta(h),s)\quad\text{for }\ (h,s)\in G\times\mathbb{R},\\
&m_\sigma((g,t),(h,s))=(gh,t)\quad\text{if }\ \alpha_\sigma(g,t)=\beta_\sigma(h,s),\\
&\epsilon_\sigma(x,t)=(\epsilon(x),t)\quad\text{for }\ (x,t)\in M\times\mathbb{R},\\
&\iota_\sigma(g,t)=(\iota(g),\sigma(g)+t)\quad\text{for }\ (g,t)\in G\times\mathbb{R}.
\end{aligned}\tag{5.23}$$

Thus, if $A(G\times\mathbb{R})$ is the Lie algebroid of $G\times\mathbb{R}$ and $X\in A_{(x,t)}(G\times\mathbb{R})$, it is clear that $X\in A_xG$ and therefore the map

$$\begin{aligned}
&\mathcal{J}: A(G\times\mathbb{R})\to AG\times\mathbb{R},\\
&X\in A_{(x,t)}(G\times\mathbb{R})\to\mathcal{J}(X)=(X,t)\in A_xG\times\mathbb{R}
\end{aligned}\tag{5.24}$$

defines an isomorphism of vector bundles. Furthermore, if on $AG\times\mathbb{R}$ we consider the Lie algebroid structure $([\![\,,\,]\!]^{-\phi_0},\bar{\rho}^{\phi_0})$ given by (2.22), then $\mathcal{J}$ is a Lie algebroid isomorphism. In conclusion, the Lie algebroid of the Lie groupoid $G\times\mathbb{R}\rightrightarrows M\times\mathbb{R}$ may be identified with $(AG\times\mathbb{R},[\![\,,\,]\!]^{-\phi_0},\bar{\rho}^{\phi_0})$.

We also have the following result.

Proposition 5.7. *Let $G\rightrightarrows M$ be a Lie groupoid and $\sigma : G\to\mathbb{R}$ be a multiplicative function. Suppose that (Λ,E) is a Jacobi structure on G, that $\tilde{\Lambda}=\mathrm{e}^{-t}(\Lambda+(\partial/\partial t)\wedge E)$ is the Poissonization on $G\times\mathbb{R}$ and that in $G\times\mathbb{R}$ we consider the Lie groupoid structure on $M\times\mathbb{R}$ with structural functions given by (5.23). Then, $(G\rightrightarrows M,\Lambda,E,\sigma)$ is a Jacobi groupoid if and only if $(G\times\mathbb{R}\rightrightarrows M\times\mathbb{R},\tilde{\Lambda})$ is a Poisson groupoid.*

Proof. From (2.10) and (5.23), it follows that the projections $(\alpha_\sigma)^{\mathrm{T}}$, $(\beta_\sigma)^{\mathrm{T}}$, the inclusion $(\epsilon_\sigma)^{\mathrm{T}}$ and the partial multiplication $\oplus_{T(G\times\mathbb{R})}$ of the tangent groupoid $T(G\times M)\rightrightarrows T(M\times\mathbb{R})$ are given by

$$\begin{aligned}
&(\alpha_\sigma)^{\mathrm{T}}\left(X_g+\lambda\frac{\partial}{\partial t|_t}\right)=\alpha^{\mathrm{T}}(X_g)+(\lambda+X_g(\sigma))\frac{\partial}{\partial t|_{t+\sigma(g)}}\\
&\qquad\qquad\text{for }\ \left(X_g+\lambda\frac{\partial}{\partial t|_t}\right)\in T_{(g,t)}(G\times\mathbb{R}),\\
&(\beta_\sigma)^{\mathrm{T}}\left(Y_h+\mu\frac{\partial}{\partial t|_s}\right)=\beta^{\mathrm{T}}(Y_h)+\mu\frac{\partial}{\partial t|_s}\quad\text{for }\ \left(Y_h+\mu\frac{\partial}{\partial t|_s}\right)\in T_{(h,s)}(G\times\mathbb{R}),\\
&\left(X_g+\lambda\frac{\partial}{\partial t|_t}\right)\oplus_{T(G\times\mathbb{R})}\left(Y_h+\mu\frac{\partial}{\partial t|_s}\right)=X_g\oplus_{TG}Y_h+\frac{\partial}{\partial t|_t},\\
&(\epsilon_\sigma)^{\mathrm{T}}\left(X_x+\lambda\frac{\partial}{\partial t|_t}\right)=\epsilon^{\mathrm{T}}(X_x)+\lambda\frac{\partial}{\partial t|_t}\quad\text{for }\ \left(X_x+\lambda\frac{\partial}{\partial t|_t}\right)\in T_{(x,t)}(M\times\mathbb{R}).
\end{aligned}\tag{5.25}$$

On the other hand, using (2.13) and (5.23), we deduce that the projections $\widetilde{\alpha_\sigma}$, $\widetilde{\beta_\sigma}$, the inclusion $\widetilde{\epsilon_\sigma}$ and the partial multiplication $\oplus_{T^*(G\times\mathbb{R})}$ in the cotangent groupoid $T^*(G\times\mathbb{R})\rightrightarrows A^*G\times\mathbb{R}$ are defined by

$$\begin{aligned}
&\widetilde{\alpha_\sigma}(\omega_g+\gamma\delta t|_t)=(\tilde\alpha(\omega_g),\sigma(g)+t)\quad \text{for }\ (\omega_g+\gamma\delta t|_t)\in T^*_{(g,t)}(G\times\mathbb{R}),\\
&\widetilde{\beta_\sigma}(\nu_h+\zeta\delta t|_s)=(\tilde\beta(\nu_h)-\zeta(\delta\sigma)_{\widetilde{\beta(g)}},s)\quad \text{for }\ (\nu_h+\zeta\delta t|_s)\in T^*_{(h,s)}(G\times\mathbb{R}),\\
&(\omega_g+\gamma\delta t|_t)\oplus_{T^*(G\times\mathbb{R})}(\nu_h+\zeta\delta t|_s)=(\omega_g+\zeta(\delta\sigma)_g)\oplus_{T^*G}\nu_h+(\gamma+\zeta)\delta t|_t,\\
&\widetilde{\epsilon_\sigma}(\omega_x,t)=\tilde\epsilon(\omega_x)+0\delta t|_t\quad \text{for }\ (\omega_x,t)\in A^*_xG\times\mathbb{R}.
\end{aligned}\tag{5.26}$$

Moreover, from (2.2), we have that the homomorphism $\#_{\tilde\Lambda}:T^*(G\times\mathbb{R})\to T(G\times\mathbb{R})$ is given by

$$\#_{\tilde\Lambda}(\omega_g+\gamma\delta t|_t)=\mathrm{e}^{-t}\left(\#_\Lambda(\omega_g)+\gamma E_g-\omega_g(E_g)\frac{\partial}{\partial t}|_t\right)\tag{5.27}$$

for $(\omega_g+\gamma\delta t|_t)\in T^*_{(g,t)}(G\times\mathbb{R})$.

Now, we consider in $T^*G\times\mathbb{R}$ (respectively, $TG\times\mathbb{R}$) the Lie groupoid structure over A^*G (respectively, $TM\times\mathbb{R}$) with structural functions defined by (3.15) (respectively, (3.8)).

Then, a straightforward computation, using (2.10), (2.13), (3.8), (3.10), (3.15), (5.25)–(5.27), shows that $\#_{(\Lambda,E)}:T^*G\times\mathbb{R}\to TG\times\mathbb{R}$ is a Lie groupoid morphism over some map $\varphi_0:A^*G\to TM\times\mathbb{R}$ if and only if $\#_{\tilde\Lambda}:T^*(G\times\mathbb{R})\to T(G\times\mathbb{R})$ is a Lie groupoid morphism over some map $\tilde\varphi_0:A^*G\times\mathbb{R}\to T(M\times\mathbb{R})$. This proves the result. □

As we know (see Section 2.4) if $((A,\phi_0),(A^*,X_0))$ is a generalized Lie bialgebroid and on the vector bundle $A\times M\to M\times\mathbb{R}$ (respectively, $A^*\times\mathbb{R}\to M\times\mathbb{R}$) we consider the Lie algebroid structure $([\![\,,]\!]^{-\phi_0},\bar\rho^{\phi_0})$ (respectively, $([\![\,,]\!]_*^{\wedge X_0},\hat\rho_*^{X_0})$), then the pair $(A\times\mathbb{R},A^*\times\mathbb{R})$ is a Lie bialgebroid. In particular, if $(G\rightrightarrows M,\Lambda,E,\sigma)$ is a Jacobi groupoid and AG is the Lie algebroid of G, then the pair $(AG\times\mathbb{R},A^*G\times\mathbb{R})$ is a Lie bialgebroid. Furthermore, we have the following proposition.

Proposition 5.8. *Let $(G\rightrightarrows M,\Lambda,E,\sigma)$ be a Jacobi groupoid and $(G\times\mathbb{R}\rightrightarrows M\times\mathbb{R},\tilde\Lambda)$ be the corresponding Poisson groupoid. If $((AG,\phi_0),(A^*G,X_0))$ (respectively, $(A(G\times\mathbb{R}),A^*(G\times\mathbb{R}))$) is the generalized Lie bialgebroid (respectively, the Lie bialgebroid) associated with $(G\rightrightarrows M,\Lambda,E,\sigma)$ (respectively, $(G\times\mathbb{R}\rightrightarrows M\times\mathbb{R},\tilde\Lambda)$), then the Lie bialgebroids $(A(G\times\mathbb{R}),A^*(G\times\mathbb{R}))$ and $(AG\times\mathbb{R},A^*G\times\mathbb{R})$ are isomorphic.*

Proof. Denote by $([\![\,,]\!],\rho)$ the Lie algebroid structure on AG and by $\mathcal{J}:A(G\times\mathbb{R})\to AG\times\mathbb{R}$ the isomorphism between the Lie algebroids $A(G\times\mathbb{R})$ and $(AG\times\mathbb{R},[\![\,,]\!]^{-\phi_0},\bar\rho^{\phi_0})$ given by (5.24).

Now, let $\tilde{\mathcal{J}}^*:T^*G\times\mathbb{R}\times\mathbb{R}\to T^*(G\times\mathbb{R})$ be the map defined by

$$\tilde{\mathcal{J}}^*(\omega_g,\gamma,t)=\omega_g+\gamma\delta t|_t\quad \text{for }\ \omega_g\in T^*_gG\ \text{ and }\ \gamma,t\in\mathbb{R}.$$

Using the results in [17] (see Section 3.2 in [17]), we deduce that

$$\tilde{\mathcal{J}}^{*}[\![(\tilde{\alpha},\tilde{f}),(\tilde{\beta},\tilde{g})]\!]^{\wedge X_0}_{(\Lambda,E)} = [\![\tilde{\mathcal{J}}^{*}(\tilde{\alpha},\tilde{f}),\tilde{\mathcal{J}}^{*}(\tilde{\beta},\tilde{g})]\!]_{\tilde{\Lambda}} = [\![\tilde{\alpha}+\tilde{f}\delta t,\tilde{\beta}+\tilde{g}\delta t]\!]_{\tilde{\Lambda}},$$
$$\#_{\tilde{\Lambda}}(\tilde{\mathcal{J}}^{*}(\tilde{\alpha},\tilde{f})) = \widehat{\tilde{\#}_{(\Lambda,E)}}^{X_0}(\tilde{\alpha},\tilde{f}) \tag{5.28}$$

for $\tilde{\alpha},\tilde{\beta}$ time-dependent 1-forms on G and $\tilde{f},\tilde{g}\in C^{\infty}(G\times\mathbb{R},\mathbb{R})$, where $([\![,]\!]_{(\Lambda,E)},\tilde{\#}_{(\Lambda,E)})$ (respectively, $([\![,]\!]_{\tilde{\Lambda}},\#_{\tilde{\Lambda}})$) is the Lie algebroid structure on $T^*G\times\mathbb{R}$ (respectively, $T^*(G\times\mathbb{R})$) induced by the Jacobi structure (Λ,E) (respectively, the Poisson structure $\tilde{\Lambda}$) on G (respectively, $G\times\mathbb{R}$).

On the other hand, if we identify A^*G (respectively, $A^*(G\times\mathbb{R})$) with the conormal bundle of $\epsilon(M)$ (respectively, $\epsilon_\sigma(M\times\mathbb{R})$), then the restriction of $\tilde{\mathcal{J}}^{*}$ to $A^*G\times\{0\}\times\mathbb{R}\cong A^*G\times\mathbb{R}$ is just the adjoint operator $\mathcal{J}^*: A^*G\times\mathbb{R}\to A^*(G\times\mathbb{R})$ of $\mathcal{J}$. Therefore, from (2.23), (5.9) and (5.28) and Remark 5.3, we conclude that the map $\mathcal{J}^*$ is an isomorphism between the Lie algebroids $(A^*G\times\mathbb{R},[\![,]\!]_*^{\wedge X_0},\hat{\rho}_*^{X_0})$ and $A^*(G\times\mathbb{R})$. □

5.3.2. Integration of generalized Lie bialgebroids

In this section, we will show a converse of Theorem 5.4.

For this purpose, we will use the notion of the derivative of an affine k-vector field on a Lie groupoid (see [31]). Let G be a Lie groupoid with Lie algebroid AG and P be an affine k-vector field on G. Then, the derivative of P, δP, is the map $\delta P:\Gamma(AG)\to\Gamma(\wedge^k(AG))$ defined as follows. If $X\in\Gamma(AG)$, $\delta P(X)$ is the element in $\Gamma(\wedge^k(AG))$ whose left-translation is $\mathcal{L}_{\overleftarrow{X}}P$.

Now, we will prove the announced result at the beginning of this section.

Theorem 5.9. *Let $((AG,\phi_0),(A^*G,X_0))$ be a generalized Lie bialgebroid where AG is the Lie algebroid of an α-connected and α-simply connected Lie groupoid $G\rightrightarrows M$. Then, there is a unique multiplicative function $\sigma: G\times\mathbb{R}$ and a unique Jacobi structure (Λ,E) on G that makes $(G\rightrightarrows M,\Lambda,E,\sigma)$ into a Jacobi groupoid with generalized Lie bialgebroid $((AG,\phi_0),(A^*G,X_0))$.*

Proof. Since G is α-connected and α-simply connected, we deduce that there exists a unique multiplicative function $\sigma: G\to\mathbb{R}$ such that

$$\phi_0(X)=X(\sigma)\quad\forall X\in\Gamma(AG).$$

The multiplicative function $\sigma: G\to\mathbb{R}$ allows us to construct a Lie groupoid structure in $G\times\mathbb{R}$ over $M\times\mathbb{R}$ with structural functions $\alpha_\sigma,\beta_\sigma,m_\sigma,\epsilon_\sigma$ and ι_σ given by (5.23).

If $([\![,]\!],\rho)$ is the Lie algebroid structure on AG then, as we know, the Lie algebroid of $G\times\mathbb{R}$ is $(AG\times\mathbb{R},[\![,]\!]^{-\phi_0},\bar{\rho}^{\phi_0})$. Moreover, if $([\![,]\!]_*,\rho_*)$ is the Lie algebroid structure on A^*G and we consider on the vector bundle $A^*G\times\mathbb{R}\to M\times\mathbb{R}$ the Lie algebroid structure $([\![,]\!]_*^{\wedge X_0},\hat{\rho}_*^{X_0})$ given by (2.23), it follows that the pair $(AG\times\mathbb{R},A^*G\times\mathbb{R})$ is a Lie bialgebroid. Therefore, using Theorem 4.1 in [31], we obtain that there is a unique Poisson structure $\tilde{\Lambda}$ on $G\times\mathbb{R}$ that makes $G\times\mathbb{R}$ into a Poisson groupoid with Lie bialgebroid $(AG\times\mathbb{R},A^*G\times\mathbb{R})$.

We will see that the 2-vector (on $G \times \mathbb{R}$) $\mathcal{L}_{\partial/\partial t}\tilde{\Lambda} + \tilde{\Lambda}$ is affine. For this purpose, we will use the following relation:

$$\mathcal{L}_{\partial/\partial t}\overleftarrow{\tilde{P}} = \frac{\overleftarrow{\partial \tilde{P}}}{\partial t} \quad \text{for} \quad \tilde{P} \in \Gamma(\wedge^k(AG \times \mathbb{R})). \tag{5.29}$$

Note that $\tilde{P}$ is a time-dependent section of the vector bundle $\wedge^k(AG) \to M$ and, thus, one may consider the derivative of $\tilde{P}$ with respect to the time, $\partial\tilde{P}/\partial t$.

From (5.29) and Proposition 4.6, we conclude that the vector field $\partial/\partial t$ is affine. Consequently (see Proposition 2.5 in [31]), the 2-vector $\mathcal{L}_{\partial/\partial t}\tilde{\Lambda} + \tilde{\Lambda}$ is also affine.

Next, we will show that the Poisson structure $\tilde{\Lambda}$ is homogeneous with respect to the vector field $\partial/\partial t$. This fact implies that $\tilde{\Lambda}$ is the Poissonization of a Jacobi structure (Λ, E) on G (see Remark 2.1). Moreover, from Propositions 5.7 and 5.8, we will have that $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid with generalized Lie bialgebroid $((AG, \phi_0), (A^*G, X_0))$.

Therefore, we must prove that $\tilde{\Lambda}$ is homogeneous. Now, using Theorem 2.6 in [31] and since G is α-connected and the 2-vector $\mathcal{L}_{\partial/\partial t}\tilde{\Lambda} + \tilde{\Lambda}$ is affine, we deduce that $\tilde{\Lambda}$ is homogeneous if and only if:

(i) the derivative of the 2-vector $\mathcal{L}_{\partial/\partial t}\tilde{\Lambda} + \tilde{\Lambda}$ is zero,
(ii) the restriction of the 2-vector $\mathcal{L}_{\partial/\partial t}\tilde{\Lambda} + \tilde{\Lambda}$ to the points of $\epsilon_\sigma(M \times \mathbb{R})$ is zero.

First, we will show (i). If H is a Poisson groupoid with Poisson structure π and Lie algebroid AH, we have that (see Theorem 3.1 in [42])

$$\mathcal{L}_{\overleftarrow{X}}\pi = -\overleftarrow{\mathrm{d}_* X} \tag{5.30}$$

for $X \in \Gamma(AH)$, where d_* is the differential of the dual Lie algebroid A^*H. Thus, from (5.29) and (5.30), it follows that

$$\mathcal{L}_{\overleftarrow{\tilde{X}}}(\mathcal{L}_{\partial/\partial t}\tilde{\Lambda} + \tilde{\Lambda}) = \mathcal{L}_{\partial/\partial t}\mathcal{L}_{\overleftarrow{\tilde{X}}}\tilde{\Lambda} - \mathcal{L}_{\overleftarrow{\partial\tilde{X}/\partial t}}\tilde{\Lambda} + \mathcal{L}_{\overleftarrow{\tilde{X}}}\tilde{\Lambda} = \overleftarrow{\hat{\mathrm{d}}_*^{X_0}\frac{\partial\tilde{X}}{\partial t}} - \overleftarrow{\hat{\mathrm{d}}_*^{X_0}\tilde{X}} - \overleftarrow{\frac{\partial(\hat{\mathrm{d}}_*^{X_0}\tilde{X})}{\partial t}}$$

for $\tilde{X} \in \Gamma(AG \times \mathbb{R})$. On the other hand, using the results in [17] (see Remark B.3 in [17]), we obtain that

$$\hat{\mathrm{d}}_*^{X_0}\tilde{Z} = \mathrm{e}^{-t}\left(\hat{\mathrm{d}}_*^0\tilde{Z} + X_0 \wedge \left(\tilde{Z} + \frac{\partial\tilde{Z}}{\partial t}\right)\right) \quad \text{for} \quad \tilde{Z} \in \Gamma(AG \times \mathbb{R}),$$

$\hat{\mathrm{d}}_*^0$ being the differential of the Lie algebroid $(A^*G \times \mathbb{R}, [\![\,,\,]\!]_*^{\wedge 0}, \hat{\rho}_*^0)$. Consequently, we deduce that

$$\mathcal{L}_{\overleftarrow{\tilde{X}}}(\mathcal{L}_{\partial/\partial t}\tilde{\Lambda} + \tilde{\Lambda}) = 0.$$

Next, we will show (ii). If (x, t) is a point of $M \times \mathbb{R}$, then

$$T^*_{\epsilon_\sigma(x,t)}(G \times \mathbb{R}) \cong A^*_{(x,t)}(G \times \mathbb{R}) \oplus ((\alpha_\sigma)_*^{(\tilde{x},t)})^*(T^*_{(x,t)}(M \times \mathbb{R})).$$

Therefore, it is enough to prove that

$$(\mathcal{L}_{\partial/\partial t}\tilde{\Lambda} + \tilde{\Lambda})(\delta F_1, \delta F_2)|_{\epsilon_\sigma(M\times\mathbb{R})} = 0,$$

when F_1 and F_2 are either constant on $\epsilon_\sigma(M\times\mathbb{R})$ or equal to $(\alpha_\sigma)^* f_i$, with $f_i \in C^\infty(M\times\mathbb{R},)$, $i=1,2$. We will distinguish three cases.

First case. Suppose that $F_1=(\alpha_\sigma)^* f_1$ and $F_2=(\alpha_\sigma)^* f_2$, with $f_1, f_2 \in C^\infty(M\times\mathbb{R},\mathbb{R})$. Denote by $\tilde{\Lambda}_0$ the Poisson structure on $M\times\mathbb{R}$ induced by the Lie bialgebroid $(AG\times\mathbb{R}, A^*G\times\mathbb{R})$ and by $\{,\}_{\tilde{\Lambda}}$ (respectively, $\{,\}_{\tilde{\Lambda}_0}$) the Poisson bracket on $G\times\mathbb{R}$ (respectively, $M\times\mathbb{R}$) associated with $\tilde{\Lambda}$ (respectively, $\tilde{\Lambda}_0$). Then, from Proposition 5.7 and since the vector field $\partial/\partial t$ on $G\times\mathbb{R}$ is α_σ-projectable, it follows that

$$\begin{aligned}&(\mathcal{L}_{\partial/\partial t}\tilde{\Lambda}+\tilde{\Lambda})(\delta F_1,\delta F_2)\\ &\quad=\alpha_\sigma^*\left(\frac{\partial}{\partial t}\{f_1,f_2\}_{\tilde{\Lambda}_0}-\left\{\frac{\partial f_1}{\partial t},f_2\right\}_{\tilde{\Lambda}_0}-\left\{f_1,\frac{\partial f_2}{\partial t}\right\}_{\tilde{\Lambda}_0}+\{f_1,f_2\}_{\tilde{\Lambda}_0}\right).\end{aligned}$$

Thus, using that the Poisson structure $\tilde{\Lambda}_0$ is homogeneous with respect to the vector field $\partial/\partial t$ on $M\times\mathbb{R}$ (see Theorem 2.13), we obtain that

$$(\mathcal{L}_{\partial/\partial t}\tilde{\Lambda}+\tilde{\Lambda})(\delta F_1,\delta F_2)=0.$$

Second case. Suppose that $F_1=(\alpha_\sigma)^* f_1$, with $f_1\in C^\infty(M\times\mathbb{R},\mathbb{R})$ and that F_2 is constant on $\epsilon_\sigma(M\times\mathbb{R})$. Following the proof of Lemma 4.12 in [31], we deduce that

$$\{(\alpha_\sigma)^* f, H\}_{\tilde{\Lambda}}=\overleftarrow{((\hat{\rho}_*^{X_0})^*(\delta f))}(H)\tag{5.31}$$

for $f\in C^\infty(M\times\mathbb{R},\mathbb{R})$ and $H\in C^\infty(G\times\mathbb{R},\mathbb{R})$. Note that (see (2.23))

$$(\hat{\rho}_*^{X_0})^*(\omega+g\delta t)=\mathrm{e}^{-t}((\rho_*)^*(\omega)+gX_0)\tag{5.32}$$

for $g\in C^\infty(M\times\mathbb{R},\mathbb{R})$ and ω a time-dependent 1-form on M. Therefore, from (5.29), (5.31) and (5.32) and since $\partial F_2/\partial t=0$, we have that

$$\begin{aligned}&(\mathcal{L}_{\partial/\partial t}\tilde{\Lambda}+\tilde{\Lambda})(\delta F_1,\delta F_2)\\ &\quad=-\left[\frac{\partial}{\partial t},\overleftarrow{(\hat{\rho}_*^{X_0})^*(\delta f_1)}\right](F_2)-\overleftarrow{(\hat{\rho}_*^{X_0})^*\left(\delta\left(\frac{\partial f_1}{\partial t}\right)\right)}(F_2)-\overleftarrow{(\hat{\rho}_*^{X_0})^*(\delta f_1)}(F_2)\\ &\quad=\overleftarrow{\frac{\partial}{\partial t}((\hat{\rho}_*^{X_0})^*(\delta f_1))}-\overleftarrow{(\hat{\rho}_*^{X_0})^*\left(\delta\left(\frac{\partial f_1}{\partial t}\right)\right)}-\overleftarrow{(\hat{\rho}_*^{X_0})^*(\delta f_1)}(F_2)=0.\end{aligned}$$

Third case. Suppose that F_1 and F_2 are constant on $\epsilon_\sigma(M\times\mathbb{R})$. Then, using that $\epsilon_\sigma(M\times\mathbb{R})$ is a coisotropic submanifold of $(G\times\mathbb{R},\tilde{\Lambda})$, it follows that

$$\{F_1,F_2\}_{\tilde{\Lambda}|_{\epsilon_\sigma(M\times\mathbb{R})}}=0.$$

Moreover, since $\partial F_1/\partial t=\partial F_2/\partial t=0$ and the restriction to $\epsilon_\sigma(M\times\mathbb{R})$ of the vector field $\partial/\partial t$ is tangent to $\epsilon_\sigma(M\times\mathbb{R})$, we conclude that

$$(\mathcal{L}_{\partial/\partial t}\tilde{\Lambda}+\tilde{\Lambda})(\delta F_1,\delta F_2)|_{\epsilon_\sigma(M\times\mathbb{R})}=0. \qquad\square$$

Example 5.10.

1. *Lie bialgebroids*

   Let (AG, A^*G) be a Lie bialgebroid where AG is the Lie algebroid of an α-connected and α-simply connected Lie groupoid $G \rightrightarrows M$. Then, using Theorem 5.9 (see also Examples 4.3,1), we obtain that there exists a unique Poisson structure Λ on G that makes $(G \rightrightarrows M, \Lambda)$ into a Poisson groupoid with Lie bialgebroid (AG, A^*G). This result was proved in [31].

2. *Generalized Lie bialgebras*

   If G is a connected Lie group with identity element $\mathfrak{e}$, $\sigma : G \to \mathbb{R}$ is a multiplicative function and Λ is a σ-multiplicative 2-vector such that the intrinsic derivative of Λ at $\mathfrak{e}$ is zero, then Λ identically vanishes (see [18]).

   Let $((\mathfrak{g}, \phi_0)(\mathfrak{g}^*, X_0))$ be a generalized Lie bialgebra, i.e. a generalized Lie bialgebroid over a single point, and G be a connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Then, using (5.9), Proposition 4.4 and Theorem 5.9 we deduce the following facts: (a) there exists a unique multiplicative function $\sigma : G \to \mathbb{R}$ and a unique σ-multiplicative 2-vector Λ on G such that $(\delta\sigma)(\mathfrak{e}) = \phi_0$ and the intrinsic derivative of Λ at $\mathfrak{e}$ is $-\mathrm{d}_{*X_0}$, d_{*X_0} being the X_0-differential of the Lie algebra $\mathfrak{g}^*$; (b) $\#_\Lambda(\delta\sigma) = \overrightarrow{X}_0 - \mathrm{e}^{-\sigma}\overleftarrow{X_0}$ and (c) the pair (Λ, E) is a Jacobi structure on G, where $E = -\overrightarrow{X_0}$. These results were proved in [18] (see Theorem 3.10 in [18]).

Acknowledgements

Research partially supported by DGICYT grant BFM 2000-0808. D. Iglesias-Ponte wishes to thank the Spanish Ministry of Education and Culture for an FPU grant.

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